Let $A, B, C, \cdots$ denote some actions, and let $\delta_A, \delta_B, \delta_C, \cdots$ be the corresponding decisions, i.e.,

$$
\delta_X = \begin{cases} 
1, & \text{if action } X \text{ is taken;} \\
0, & \text{otherwise.}
\end{cases}
$$

We express logical conditions on the actions by constraints on the corresponding decision variables.

1. **And.** The logical condition

   $A$ and $B$

   is equivalent to the constraint

   $$
   \delta_A + \delta_B = 2
   $$

   This equivalence is expressed as follows

   $$
   A \text{ and } B \iff \delta_A + \delta_B = 2
   $$

   and can be checked by a truth table,

   \begin{array}{|c|c|c|c|}
   \hline
   \delta_A & \delta_B & \text{A and B} & \delta_A + \delta_B \\
   \hline
   0 & 0 & \text{F} & 0 \\
   0 & 1 & \text{F} & 1 \\
   1 & 0 & \text{F} & 1 \\
   1 & 1 & \text{T} & 2 \\
   \hline
   \end{array}

2. **Or.**

   $A$ or $B$ or both \iff $\delta_A + \delta_B \geq 1$

   \begin{array}{|c|c|c|c|}
   \hline
   \delta_A & \delta_B & \text{A or B} & \delta_A + \delta_B \\
   \hline
   0 & 0 & \text{F} & 0 \\
   0 & 1 & \text{T} & 1 \\
   1 & 0 & \text{T} & 1 \\
   1 & 1 & \text{T} & 2 \\
   \hline
   \end{array}

3. **Exclusive Or.**

   $A$ or $B$ but not both \iff $\delta_A + \delta_B = 1$

   \begin{array}{|c|c|c|c|}
   \hline
   \delta_A & \delta_B & \text{A or B but not both} & \delta_A + \delta_B \\
   \hline
   0 & 0 & \text{F} & 0 \\
   0 & 1 & \text{T} & 1 \\
   1 & 0 & \text{T} & 1 \\
   1 & 1 & \text{F} & 2 \\
   \hline
   \end{array}
4. If $A$ then $B$.
Synonyms: $B$ if $A$, $A$ only if $B$, $A$ implies $B$, $A$ is sufficient for $B$, $B$ is necessary for $A$, $A \implies B$ 

$B$ if $A \iff \delta_A \leq \delta_B$

<table>
<thead>
<tr>
<th>$\delta_A$</th>
<th>$\delta_B$</th>
<th>$B$ if $A$</th>
<th>$\delta_A \leq \delta_B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>T</td>
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<tr>
<td>1</td>
<td>0</td>
<td>F</td>
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<tr>
<td>1</td>
<td>1</td>
<td>T</td>
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</tr>
</tbody>
</table>

5. If $B$ then $A$.
Synonyms: $A$ if $B$, $A$ only if $B$, $B$ implies $A$, $A$ is necessary for $B$, $A \iff B$ 

$B$ only if $A \iff \delta_A \geq \delta_B$

<table>
<thead>
<tr>
<th>$\delta_A$</th>
<th>$\delta_B$</th>
<th>$B$ only if $A$</th>
<th>$\delta_A \geq \delta_B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
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</table>

6. $A$ if and only if $B$.
Synonyms: $A$ and $B$ are equivalent, $A$ is necessary and sufficient for $B$, $A \iff B$

$A$ if and only if $B \iff \delta_A = \delta_B$

<table>
<thead>
<tr>
<th>$\delta_A$</th>
<th>$\delta_B$</th>
<th>$A$ if and only if $B$</th>
<th>$\delta_A = \delta_B$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>T</td>
<td>T</td>
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</tbody>
</table>

7. If $(A$ and $B)$ then $C$.

If $(A$ and $B)$ then $C \iff \delta_A + \delta_B \leq 1 + \delta_C$

<table>
<thead>
<tr>
<th>$\delta_A$</th>
<th>$\delta_B$</th>
<th>$\delta_C$</th>
<th>If $(A$ and $B)$ then $C$</th>
<th>$\delta_A + \delta_B \leq 1 + \delta_C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
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</table>
8. If \((A \text{ or } B)\) then \(C\).

If \((A \text{ or } B)\) then \(C \iff \delta_A + \delta_B \leq 2 \delta_C\)

<table>
<thead>
<tr>
<th>(\delta_A)</th>
<th>(\delta_B)</th>
<th>(\delta_C)</th>
<th>If ((A \text{ or } B)) then (C)</th>
<th>(\delta_A + \delta_B \leq 2 \delta_C)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>T</td>
<td>T</td>
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</tbody>
</table>

9. Fixed cost. Let \(X\) be a quantity to be produced (or purchased), and let \(\delta_X\) be the decision to produce or purchase. Let \(K\) be a fixed cost, and \(c\) a variable unit cost, so the cost function has two terms

\[ K \delta_X + c X \]

To make sure that \(X > 0 \implies \delta_X = 1\) we add the constraint

\[ 0 \leq X \leq U \delta_X \]

where \(U\) is some upper bound on \(X\).

10. Quantity bounds. Let \(X\) be a quantity to be produced (or purchased), and assume that if \(X > 0\) then \(X\) must satisfy

\[ L \leq X \leq U \]

where the bounds \(L, U\) are given. To enforce this constraint we use the logical variable \(\delta_X\), the decision to produce or purchase. The quantity bounds are enforced by the constraints

\[ L \delta_X \leq X \leq U \delta_X \]

11. Quantity discounts–A. This is the case where the cost per unit decreases as the order size increases, so that “big buyers” pay less per unit.

A “fixed cost” as in § 9 results in a quantity discount. Consider for example a fixed cost \(K = \$1000\), and a variable unit cost \(c = \$20\), and let \(n\) units be bought. Then the cost for unit is

\[ \frac{1000 + 20n}{n} \]

as illustrated in the following table

<table>
<thead>
<tr>
<th>Order size (n)</th>
<th>Cost per unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1,020.00</td>
</tr>
<tr>
<td>10</td>
<td>120.00</td>
</tr>
<tr>
<td>100</td>
<td>30.00</td>
</tr>
<tr>
<td>1,000</td>
<td>21.00</td>
</tr>
<tr>
<td>10,000</td>
<td>20.10</td>
</tr>
<tr>
<td>100,000</td>
<td>20.01</td>
</tr>
</tbody>
</table>

For large \(n\) the fixed cost becomes negligible – an advantage of “big business”.

12. Quantity discounts–B. Another type of quantity discount uses price steps. For example, let the price per unit \( c \) be given as a function of the order size \( x \) as follows,

\[
c = \begin{cases} 
10, & \text{if } x \leq 1000; \\
9, & \text{if } 1001 \leq x \leq 2000; \\
8, & \text{if } 2001 \leq x.
\end{cases}
\]

and suppose we know an upper bound of \( U = 5,000 \) on the number of units we may buy. We use the variables

- \( x_1 \) = the number of pieces bought at the regular price,
- \( x_2 \) = the number of pieces bought at the price of 9 $/unit,
- \( x_3 \) = the number of pieces bought at the price of 8 $/unit.

and three binary variables,

\[
\delta_i = \begin{cases} 
1, & \text{if } x_i > 0; \\
0, & \text{otherwise.}
\end{cases}
\]

The quantity we buy is

\[
x_1 + x_2 + x_3 \quad \text{(at most one of them is non-zero)}
\]

at the cost

\[
10x_1 + 9x_2 + 8x_3
\]

with constraints

\[
egin{align*}
0 & \leq x_1 \leq 1000 \delta_1 \\
1001 \delta_2 & \leq x_2 \leq 2000 \delta_2 \\
2001 \delta_3 & \leq x_3 \leq 5000 \delta_3 \\
\delta_1 + \delta_2 + \delta_3 & \leq 1
\end{align*}
\]

Note: The last constraint allows not buying anything. If we had to buy something, this constraint would be \( = 1 \).

13. Quantity discounts–C. Another kind of quantity discount is illustrated by the following unit price

\[
c = \begin{cases} 
10, & \text{for the first 1000 units;} \\
9, & \text{for any unit above 1000, and up to 2000;} \\
8, & \text{for any unit above 2000.}
\end{cases}
\]

and assume an upper bound of 5,000 units.

We use the variables

- \( x_1 \) = the number of pieces bought at the regular price,
- \( x_2 \) = the number of pieces bought at the price of 9 $/unit,
- \( x_3 \) = the number of pieces bought at the price of 8 $/unit.

and three binary variables,

\[
\delta_i = \begin{cases} 
1, & \text{if } x_i > 0; \\
0, & \text{otherwise.}
\end{cases}
\]

The quantity we buy is

\[
x_1 + x_2 + x_3 \quad \text{(now all of them may be non-zero)}
\]

at the cost

\[
10x_1 + 9x_2 + 8x_3
\]
with the constraints

\[ 1000 \delta_2 \leq x_1 \leq 1000 \delta_1 \]
\[ 1000 \delta_3 \leq x_2 \leq 1000 \delta_2 \]
\[ 0 \leq x_3 \leq 3000 \delta_3 \], (remember the upper bound of 5,000 units)
\[ \delta_1 \geq \delta_2 \]
\[ \delta_2 \geq \delta_3 \]

that guarantee, for example, that we cannot use the discounted unit price of $9 before buying 1,000 units at the regular price, etc.