LOGICAL CONSTRAINTS AND BINARY VARIABLES

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Let $A, B, C, \cdots$ denote some actions, and let δ_A, δ_B, $\delta_C, \cdots$ be the corresponding decisions, i.e.,

$$\delta_X = \begin{cases} 1, & \text{if action } X \text{ is taken;} \\ 0, & \text{otherwise.} \end{cases}$$

We express logical conditions on the actions by constraints on the corresponding decision variables.

1. And. The logical condition

$$A \text{ and } B$$

is equivalent to the constraint

$$\delta_A + \delta_B = 2$$

This equivalence is expressed as follows

$$A \text{ and } B \iff \delta_A + \delta_B = 2$$

and can be checked by a truth table,

δ_A	δ_B	A and B	$\delta_A + \delta_B$
0	0	F	0
0	1	F	1
1	0	F	1
1	1	T	2

2. Or.

$$A \text{ or } B \text{ or both} \iff \delta_A + \delta_B \geq 1$$

δ_A	δ_B	A or B	$\delta_A + \delta_B$
0	0	F	0
0	1	T	1
1	0	T	1
1	1	T	2

3. Exclusive Or.

$$A \text{ or } B \text{ but not both} \iff \delta_A + \delta_B = 1$$

δ_A	δ_B	A or B but not both	$\delta_A + \delta_B$
0	0	F	0
0	1	T	1
1	0	T	1
1	1	F	2

4. If A then B.
Synonyms: B if A, A only if B, A implies B, A is sufficient for B, B is necessary for A, $A \Longrightarrow B$

$$B \text{ if } A \iff \delta_A \leq \delta_B$$

δ_A	δ_B	B if A	$\delta_A \leq \delta_B$
0	0	T	T
0	1	T	T
1	0	F	F
1	1	T	T

5. If B then A.
Synonyms: A if B, B only if A, B implies A, B is sufficient for A, A is necessary for B, $A \Longleftarrow B$

$$B \text{ only if } A \iff \delta_A \geq \delta_B$$

δ_A	δ_B	B only if A	$\delta_A \geq \delta_B$
0	0	T	T
0	1	F	F
1	0	T	T
1	1	T	T

6. A if and only if B.
Synonyms: A and B are equivalent, A is necessary and sufficient for B, $A \iff B$

$$A \text{ if and only if } B \iff \delta_A = \delta_B$$

δ_A	δ_B	A if and only if B	$\delta_A = \delta_B$
0	0	T	T
0	1	F	F
1	0	F	F
1	1	T	T

7. If (A and B) then C.

$$\text{If } (A \text{ and } B) \text{ then } C \iff \delta_A + \delta_B \leq 1 + \delta_C$$

δ_A	δ_B	δ_C	If (A and B) then C	$\delta_A + \delta_B \leq 1 + \delta_C$
0	0	0	T	T
0	0	1	T	T
0	1	0	T	T
0	1	1	T	T
1	0	0	T	T
1	0	1	T	T
1	1	0	F	F
1	1	1	T	T

8. If (A or B) then C.

$$\text{If } (A \text{ or } B) \text{ then } C \iff \delta_A + \delta_B \leq 2\,\delta_C$$

δ_A	δ_B	δ_C	If (A or B) then C	$\delta_A + \delta_B \leq 2\,\delta_C$
0	0	0	T	T
0	0	1	T	T
0	1	0	F	F
0	1	1	T	T
1	0	0	F	F
1	0	1	T	T
1	1	0	F	F
1	1	1	T	T

9. Fixed cost. Let X be a quantity to be produced (or purchased), and let δ_X be the decision to produce or purchase. Let K be a fixed cost, and c a variable unit cost, so the cost function has two terms

$$K\,\delta_X + c\,X$$

To make sure that $X > 0 \implies \delta_X = 1$ we add the constraint

$$0 \leq X \leq U\,\delta_X$$

where U is some upper bound on X.

10. Quantity bounds. Let X be a quantity to be produced (or purchased), and assume that if $X > 0$ then X must satisfy

$$L \leq X \leq U$$

where the bounds L, U are given. To enforce this constraint we use the logical variable δ_X, the decision to produce or purchase. The quantity bounds are enforced by the constraints

$$L\,\delta_X \leq X \leq U\,\delta_X$$

11. Quantity discounts–A. This is the case where the cost per unit decreases as the order size increases, so that "big buyers" pay less per unit.

A "fixed cost" as in § 9 results in a quantity discount. Consider for example a fixed cost $K =$ \$1000, and a variable unit cost $c =$ \$20, and let n units be bought. Then the cost for unit is

$$\frac{1000 + 20\,n}{n}$$

as illustrated in the following table

Order size n	Cost per unit
1	1,020.00
10	120.00
100	30.00
1,000	21.00
10,000	20.10
100,000	20.01

For large n the fixed cost becomes negligible – an advantage of "big business".

12. Quantity discounts–B. Another type of quantity discount uses price steps. For example, let the price per unit c be given as a function of the order size x as follows,

$$c = \begin{cases} 10, & \text{if } x \le 1000; \\ 9, & \text{if } 1001 \le x \le 2000; \\ 8, & \text{if } 2001 \le x. \end{cases}$$

and suppose we know an upper bound of $U = 5,000$ on the number of units we may buy. We use the variables
x_1 = the number of pieces bought at the regular price,
x_2 = the number of pieces bought at the price of 9 \$/unit,
x_3 = the number of pieces bought at the price of 8 \$/unit.
and three binary variables,

$$\delta_i = \begin{cases} 1, & \text{if } x_i > 0; \\ 0, & \text{otherwise.} \end{cases}$$

The quantity we buy is

$$x_1 + x_2 + x_3 \quad \text{(at most one of them is non-zero)}$$

at the cost

$$10\,x_1 + 9\,x_2 + 8\,x_3$$

with constraints

$$\begin{aligned} 0 \le x_1 &\le 1000\,\delta_1 \\ 1001\,\delta_2 \le x_2 &\le 2000\,\delta_2 \\ 2001\,\delta_3 \le x_3 &\le 5000\,\delta_3 \\ \delta_1 + \delta_2 + \delta_3 &\le 1 \end{aligned}$$

Note: The last constraint allows not buying anything. If we <u>had</u> to buy something, this constraint would be $= 1$.

13. Quantity discounts–C. Another kind of quantity discount is illustrated by the following unit price

$$c = \begin{cases} 10, & \text{for the first 1000 units;} \\ 9, & \text{for any unit above 1000, and up to 2000;} \\ 8, & \text{for any unit above 2000.} \end{cases}$$

and assume an upper bound of $5,000$ units.
We use the variables
x_1 = the number of pieces bought at the regular price,
x_2 = the number of pieces bought at the price of 9 \$/unit,
x_3 = the number of pieces bought at the price of 8 \$/unit.
and three binary variables,

$$\delta_i = \begin{cases} 1, & \text{if } x_i > 0; \\ 0, & \text{otherwise.} \end{cases}$$

The quantity we buy is

$$x_1 + x_2 + x_3 \quad \text{(now all of them may be non-zero)}$$

at the cost

$$10\,x_1 + 9\,x_2 + 8\,x_3$$

with the constraints

$$
\begin{aligned}
1000\,\delta_2 \le x_1 &\le 1000\,\delta_1 \\
1000\,\delta_3 \le x_2 &\le 1000\,\delta_2 \\
0 \le x_3 &\le 3000\,\delta_3\ ,\ \text{(remember the upper bound of 5,000 units)} \\
\delta_1 &\ge \delta_2 \\
\delta_2 &\ge \delta_3
\end{aligned}
$$

that guarantee, for example, that we cannot use the discounted unit price of \$9 before buying 1,000 units at the regular price, etc.