Dynamic Pricing in Airline Seat Management for Flights with Multiple Flight Legs

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Consider a multiple booking class airline-seat inventory control problem that relates to either a single flight leg or to multiple flight legs. During the time before the flight, the airline may face the problems of (1) what are the suitable prices for the opened booking classes, and (2) when to close those opened booking classes. This work deals with these two problems by only using the pricing policy. In this paper, a dynamic pricing model is developed in which the demand for tickets is modeled as a discrete time stochastic process. An important result of this work is that the strategy for the ticket booking policy can be reduced to sets of critical decision periods, which eliminates the need for large amounts of data storage.

Airline companies' revenues come mostly from seat ticket sales. Therefore, it is natural that each company attempts to establish an optimal ticket selling policy to attain the maximum possible revenue for each flight, in the sense stated below. A pool of identical seats is usually divided into several booking classes, each having different ticket prices. For example, in Japan Airlines, all seats in the economy class are divided into at least five booking classes: coach cabin without discount, coach cabin with a light discount, coach cabin with a moderate discount, coach cabin with a special discount, and coach cabin with a substantial discount. Other companies adopt special plans to serve customers with special food entrees, discount coupons, and so on.

For the case in which too many low price tickets are sold, the airline company may run the risk of losing the revenue from customers who would be willing to pay for higher price tickets later on. In contrast, if they reject too many customers who request the lower booking classes, they run the risk that the airplane will take off with many vacant seats.

Taken together, these facts imply that, to avoid such risks and thus attain the maximum expected revenue for all flights, it is important to establish an optimal booking policy for the ticket sales.

This problem has been investigated over the years using various mathematical models. In general, the previous work on this problem can be categorized as belonging to one of two basic model types, static or dynamic. Readers interested in past work can refer to the two review papers by BELOBABA (1987) and WEATHERFORD and BODILY (1992).

In a static model, the booking period is regarded as a single interval, and the problem is to set a booking limit for every booking class at the start of the booking process. A weakness of this approach is its inability to consider the actual current booking status during the process. However, it can readily handle large problems and can address the multiple flight-leg problem.

A dynamic model sets the booking limit for each booking class according to the actual bookings throughout the entire booking process. A weakness of the dynamic approach is that it is computationally intensive. The present paper uses a dynamic model.

A simple approach for the problem is to control each flight leg independently. By using the dynamic programming approach, ALSTRUP et al. (1986) deal with an overbooking problem with two types of passengers. GERCHAK, PARLAR, and YEE (1985) deal with a dynamic model of two demand classes in which demands are modeled as a discrete time stochastic process. The assumption that the demands are stochastic eliminates the need for the additional

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assumption that the demand from the different booking classes arrives sequentially. This latter assumption is required in many existing models (e.g., BRUMELLE et al., 1990; BRUMELLE and MCGILL, 1993; CURRY, 1990; ROBINSON, 1995; SAWAKI, 1989; and WOLLMER, 1992). An important result in GERCHAK et al. (1985) is that the booking policy parameters can be reduced to two types of critical values. One is the critical booking capacity and the other is critical decision periods. These values play an important role in reducing the computational time and eliminating the need for a large amount of data storage. In an extension of the work in GERCHAK et al. (1985), LEE and HERSH (1993) develop a dynamic model with multiple booking classes and multiple seat bookings. In the present paper, demands are also modeled as a discrete-time stochastic process.

Because seat inventory control may not be treated optimally in a single flight-leg control strategy, the multiple flight-leg problem cannot be ignored when trying to obtain the largest possible revenue. Among the models for the multiple flight-leg problem, HERSH and LADANY (1978) also use the dynamic program to develop a one intermediate stop (i.e. two flight legs) dynamic model under the assumption that no boarding rights are established at the intermediate stop. Moreover, CURRY (1990) uses linear programming to determine allocations for each booking class when demands are assumed to follow a continuous probability distribution, and fare classes are based on origin–destination.

DROR, TRUDEAU, and LADANY (1988) use mathematical programming to deal with a network static model assuming deterministic demands. WONG, KOPPELMAN, and DASKIN (1993) have discussed a single booking class, multiple flight-leg problem.

It is to be noted in the above dynamic modeling work that no one dealt with the multiple flight leg with multiple booking class problem except for Hersh and Ladany (1978). However, Hersh and Ladany do not mathematically analyze the structure of the booking policy, and so there is no discussion of critical value derivation. One of the objectives of the present paper is to simplify a multiple booking class, multiple flight-leg booking policy to sets of critical values.

The models in the above mentioned literature do not take into consideration the related problem of dynamic pricing. However, dynamic pricing has been examined by FENG and GALLEGO (1995) and GALLEGO and VAN RYZIN (1994, 1997). Feng and Gallego (1995) develop optimal threshold rules when demands are modeled as a continuous time stochastic process. Under the assumption that customers do not adjust their purchasing behavior in response to a firm's pricing policy, Gallego and van Ryzin (1997) deal with a multiproduct dynamic pricing model, which can be applied to network flights, and Gallego and van Ryzin (1994) deal with the strategy for changing prices.

By incorporating a cumulative distribution function of the maximum permissible purchasing price, this work deals with a discrete time dynamic pricing model in which the airline can set its prices at the time of booking request arrivals. The pricing policy will include suitable prices for the opened booking classes with respect to different combination of booking statues and remaining planning time, and when to close those opened booking classes.

The rest of the paper is organized as follows. In Section 1, we introduce notations and state our assumptions. In Section 2, we formulate the problem as a dynamic programming model from which the problem can be immediately derived for any number of flight legs. In Section 3, we show the results of the mathematical analysis. From our analysis, we find that booking policy can be reduced to sets of critical booking capacities in the cases of one or two flight legs without multiple seat bookings. Also, in the case of any number of flight legs with multiple seat bookings, the booking policy can be reduced to sets of critical decision periods. In Section 4, we demonstrate the properties of our model by some numerical examples. Finally, we present some future work in Section 5.

1. NOTATION AND ASSUMPTIONS

FIRST, FOR CONVENIENCE, we divide the total planning horizon into T decision periods small enough that no more than one customer arrives per period. Also, we number our decision periods in reverse sequence, i.e., $t = 1$ will refer to the final decision period, $t = 2$ to the period before the final decision period, and so on. By an abuse of notation, we will also use t for the number of periods remaining.

Suppose there exist $N + 1$ airports and let them be numbered as $0, 1, \ldots, N$, respectively, where flights visit each airport along a chain in sequence (Fig. 1). At the end of the period t_j, a flight will depart from the airport $j - 1$ to the next airport j, where $j = 1, 2, \ldots, N$ and $T > t_1 > t_2 > \cdots > t_N = 1$. Then, the airports $0, 1, \ldots, N$ are connected by N flights. Let a trip from a starting airport j to a destination airport k be composed of subsequent nonstop trips $(j, j + 1)$, $(j + 1, j + 2)$, $\ldots$, $(k - 1, k)$. In other words, a trip from an airport j to a destination airport k is via airports $j + 1, \ldots, k - 1$ and we will denote this trip as the trip (j, k).

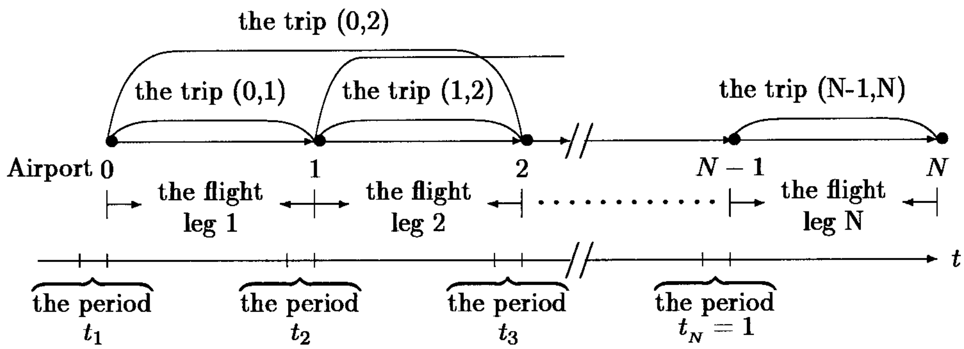

Fig. 1. Sequential airline network.

Furthermore, we assume each trip (j, k) is divided into L^{jk} booking classes.

The airline company that owns those flights tries to set its strategy for selling seats on those flights within $T > t_1$ decision periods to maximize its total expected revenue.

We are interested in maximizing the airline's total expected revenues by determining the optimal price with respect to different combinations of remaining decision periods and seats available. The pricing policy will include the suitable price for each opened booking class and when to close those opened booking classes. Now let us state the following assumptions and introduce the following notation.

ASSUMPTIONS.

1. *Cancellations, no-shows, and overbooking are not considered.*
2. *A request for multiple seat bookings is either satisfied or denied in its entirety.*
3. *A rejected customer's request for a later flight is not considered.*
4. *Airports are visited sequentially.*

Assumption 1 is also assumed by many other authors (e.g., Curry, 1990; Lee and Hersh, 1993; Brumelle et al., 1990; and Robinson, 1995). In reality, this assumption is restrictive. In their paper, Brumelle et al. also discuss the limitation of Assumption 1. We agree with them that the analysis of this simplified version can serve as a basis for approximate solutions to more realistic versions.

NOTATION.

T = total number of decision periods.

(j, k) = a trip from airport j to airport k via airports $j + 1, j + 2, \ldots, k - 1$.

t_j = the decision period immediately before the departure of the flight departing from airport $j - 1$ to airport j.

L^{jk} = the total number of booking classes in trip (j, k).

I_j = the maximum booking capacity of the flight from airports $j - 1$ to j, $j = 1, 2, \ldots, N$.

i_j = the seats available for the flight from airports $j - 1$ to j at the beginning of a decision period (initially, $i_j = I_j$).

$\mathbf{i}$ = a vector whose elements are the number of seats available for all flights at the beginning of a decision period. (i.e., $\mathbf{i} = \{i_1, i_2, \ldots, i_N\}$).

$s^{jk}(\mathbf{i})$ = the seats available for the trip (j, k) at the beginning of a decision period (i.e., $s^{jk}(\mathbf{i}) = \min\{i_{j+1}, i_{j+2}, \ldots, i_k\}$).

$\lambda^{jk}_{t\ell}$ = the probability that a request for booking class ℓ in trip (j, k), $\ell = 1, 2, \ldots, L^{jk}$ will arrive during a decision period t.

$\theta^{jk}_{t\ell m}$ = the probability that a request for booking class ℓ in trip (j, k) in decision period t is for m seats.

M^{jk}_{ℓ} = the maximum number of seats allowed to be booked for each request.

c^{jk}_{ℓ} = a fixed cost incurred by the airline in carrying a customer in booking class ℓ in trip (j, k).

From the model definition, it follows that $\lambda^{jk}_{t\ell} = 0$ for $t < t_j$, $\lambda^{jk}_{t\ell} \geqslant 0$ for $t \geqslant t_j$, and

$$\sum_{j=0}^{N-1} \sum_{k=j+1}^{N} \sum_{\ell=1}^{L^{jk}} \lambda^{jk}_{t\ell} \leqslant 1 \quad \text{for all } t.$$

It also follows that $\theta^{jk}_{t\ell m} = 0$ for $t < t_j$ and

$$\sum_{m=1}^{M^{jk}_{\ell}} \theta^{jk}_{t\ell m} \leqslant 1 \quad \text{for} \quad t \geqslant t_j.$$

It is reasonable to assume that all customers will not purchase tickets and leave the booking system if a price higher than or equal to b^{jk}_{ℓ}, called the null price, is set. For a booking class ℓ in trip (j, k), we assume that the airline previously specified a set of price points, A_ℓ^{jk}, called the allowable price set. The highest price in each set A_ℓ^{jk} is b_ℓ^{jk}, called the null price. The null price b_ℓ^{jk} allows us to model the rejecting condition; that is, setting the price at b_ℓ^{jk} is equivalent to rejecting any customer who requests booking class ℓ in trip (j, k) (i.e., closing the booking class ℓ in trip (j, k) if the price b_ℓ^{jk} is set).

Let w be the maximum permissible price for a customer requesting booking class ℓ in trip (j, k); that is, a customer who requests booking class ℓ in trip (j, k) is willing to buy tickets if and only if the seat price is lower than w. Moreover, let w be an independent and identical random variable having a known cumulative distribution function $F_\ell^{jk}(w)$. Assume that the cumulative distribution function $F_\ell^{jk}(w)$ is strictly increasing for $w \leqslant b_\ell^{jk}$, $F_\ell^{jk}(w) \leqslant 1$ for $w \leqslant b_\ell^{jk}$, and $F_\ell^{jk}(w) = 1$ for $w \geqslant b_\ell^{jk}$. Then, the probability that a customer who requests booking class ℓ in trip (j, k) will buy tickets, provided that the price $x \in A_\ell^{jk}$ is set, can be expressed as $p_\ell^{jk}(x) = 1 - F_\ell^{jk}(x)$. For convenience, let $\hat{b}_\ell^{jk} = \max\{x | p_\ell^{jk}(x) > 0, x \in A_\ell^{jk}\}$. Finally, we assume that the fixed cost c_ℓ^{jk} is less than $\hat{b}_\ell^{jk}$.

The objective is to maximize the total expected revenue from the sales of tickets of the given number of flights by finding the optimal booking policy for any combinations of seats on hand and for the time remaining before the flights.

2. OPTIMAL EQUATION

IN THIS SECTION, we will formulate the problem as a dynamic program. First, we will consider the following two functions:

- $v_t(\mathbf{i})$ — the maximum expected additional revenue when there are t decision periods and the seat vector $\mathbf{i}$ remaining.
- $g_{t\ell m}^{jk}(\mathbf{i}, x)$ — the maximum expected additional revenue given that the seat vector $\mathbf{i}$ remains, the price $x \in A_\ell^{jk}$ is set for m seats for a booking class ℓ in trip (j, k) in period t, and a customer who requests for a booking class ℓ in trip (j, k) arrives in period t.

Then, clearly we have $v_0(\mathbf{i}) = 0$. If no customer arrives in period t, [this has the probability $(1 - \sum_{j=0}^{N-1} \sum_{k=j+1}^{N} \sum_{\ell=1}^{L^{jk}} \lambda_{t\ell}^{jk})$], the expected additional revenue is $v_{t-1}(\mathbf{i})$. If a customer arrives who requests m seats in booking class ℓ in trip (j, k), [this has the probability $\lambda_{t\ell}^{jk}\ \theta_{t\ell m}^{jk}$], two cases need to be considered: 1. the number of seats remaining for trip (j, k) is less than m; that is $s_\ell^{jk}(\mathbf{i}) < m$; and 2. the number of seats remaining for trip (j, k) is not less than m; that is $s_\ell^{jk}(\mathbf{i}) \geqslant m$. In the first case, the request will be rejected because the seats remaining cannot satisfy the customer's request. In this case, the expected additional revenue is given by $v_{t-1}(\mathbf{i})$. In the second case, suppose the price x is set for m seats for the booking class ℓ in trip (j, k), then the expected additional revenue $g_{t\ell m}^{jk}(\mathbf{i}, x)$ is given by

$$
\begin{aligned}
g_{t\ell m}^{jk}(\mathbf{i}, x) &= p_\ell^{jk}(x)(mx - mc_\ell^{jk} + v_{t-1}(\mathbf{i} - me_{jk})) \\
&\quad + (1 - p_\ell^{jk}(x))v_{t-1}(\mathbf{i}) \\
&= v_{t-1}(\mathbf{i}) + mp_\ell^{jk}(x)(x - c_\ell^{jk} - (v_{t-1}(\mathbf{i}) \\
&\quad - v_{t-1}(\mathbf{i} - me_{jk}))/m), \quad t \geqslant 1 \qquad (1)
\end{aligned}
$$

where

$$
e_{0k} = (\underset{1}{1},\ \underset{2}{1},\ \underset{\cdots}{\ldots},\ \underset{k}{1},\ \underset{k+1}{0},\ \underset{\cdots}{\ldots},\ \underset{N}{0}),
$$

$$
e_{jk} = (\underset{1}{0},\ \underset{2}{0},\ \underset{\cdots}{\ldots},\ \underset{j}{0},\ \underset{j+1}{1},\ \underset{\cdots}{\ldots},\ \underset{k}{1},\ \underset{k+1}{0},\ \underset{\cdots}{\ldots},\ \underset{N}{0}), \quad j \geqslant 1,
$$

$$
\mathbf{i} - e_{jk} = (i_1, \ldots, i_j - 1, i_{j+1} - 1, \ldots, i_k - 1, i_{k+1}, \ldots, i_N).
$$

In Eq. 1, if the price $x = b_\ell^{jk}$ is set, $g_{t\ell m}^{jk}(\mathbf{i}, x) = v_{t-1}(\mathbf{i})$. This is equivalent to rejecting a customer who requests for m seats for booking class ℓ in trip (j, k). If a price $x \in A_\ell^{jk}$ other than b_ℓ^{jkj} is set, the probability of the arriving customer purchasing tickets is $p_\ell^{jk}(x)$, and the probability of the customer leaving the booking system is $1 - p_\ell^{jk}(x)$. Then, the expected additional revenue is $mx - mc_\ell^{jk} + v_{t-1}(\mathbf{i} - me_\ell^{jk})$ and $v_{t-1}(\mathbf{i})$, respectively. Now, we define a function $I(\cdot)$ such that, if a statement S is true, then $I(S) = 1$, or else $I(S) = 0$. Then for $t \geqslant 1$ we have the backward recursive equation,

$$
\begin{aligned}
v_t(\mathbf{i}) &= \left(1 - \sum_{j=0}^{N-1} \sum_{k=j+1}^{N} \sum_{\ell=1}^{L^{jk}} \lambda_{t\ell}^{jk}\right) v_{t-1}(\mathbf{i}) \\
&\quad + \sum_{j=0}^{N-1} \sum_{k=j+1}^{N} \sum_{\ell=1}^{L^{jk}} \lambda_{t\ell}^{jk} \sum_{m=1}^{M_\ell^{jk}} \theta_{t\ell m}^{jk}(v_{t-1}(\mathbf{i}) I(s^{jk}(\mathbf{i}) < m) \\
&\quad + \max_{x \in A_\ell^{jk}} g_{t\ell m}^{jk}(\mathbf{i}, x) I(s^{jk}(\mathbf{i}) \geqslant m)), \quad t \geqslant 1. \qquad (2)
\end{aligned}
$$

Let $z_{tm}^{jk}(\mathbf{i})$ be the average maximal value per seat sold if m seats are sold in trip (j, k) in period t, that is

$$z^{jk}_{tm}(\mathbf{i}) = \frac{v_t(\mathbf{i}) - v_t(\mathbf{i} - me_{jk})}{m}, \quad s^{jk}(\mathbf{i}) \geqslant m, \quad t \geqslant 0. \tag{3}$$

Then, the expression

$$\max_{x \in A^{jk}_\ell} g^{jk}_{t\ell m}(\mathbf{i}, x)$$

in Eq. 2 can be rewritten as

$$\max_{x \in A^{jk}_\ell} g^{jk}_{t\ell m}(\mathbf{i}, x) = v_{t-1}(\mathbf{i}) + m \max_{x \in A^{jk}_\ell} p^{jk}_\ell(x)(x - z^{jk}_{t-1,m}(\mathbf{i}) - c^{jk}_\ell), \quad t \geqslant 1. \tag{4}$$

Now, for any given ℓ and $\mathbf{i}$ such that $s^{jk}(\mathbf{i}) \geqslant m$, let $x^{jk}_{t\ell m}(\mathbf{i})$ be the optimal price for m seats for the booking class ℓ in trip (j, k) in decision period t, then we can see from Eq. 4 that $x^{jk}_{t\ell m}(\mathbf{i})$ is given by the x maximizing $p^{jk}_\ell(x)(x - z^{jk}_{t-1,m}(\mathbf{i}) - c^{jk}_\ell)$. Here, we should note that $x^{jk}_{t\ell m}(\mathbf{i})$ depends on the value of $z^{jk}_{t-1,m}(\mathbf{i}) + c^{jk}_\ell$. It is convenient for us to consider the function,

$$K^{jk}_\ell(\nu) = \max_{x \in A^{jk}_\ell} p^{jk}_\ell(x)(x - \nu), \tag{5}$$

for any real number ν and let $x^{jk}_\ell(\nu)$ denote the smallest x attaining the maximum of the right hand side in Eq. 5. Then, if we know the relation between $x^{jk}_\ell(\nu)$ and ν, the relation between $x^{jk}_{t\ell m}(\mathbf{i})$ and $z^{jk}_{t-1,m}(\mathbf{i}) + c^{jk}_\ell$ can be derived. Moreover, by Eq. 5, $v_t(\mathbf{i})$ can be rewritten as

$$v_t(\mathbf{i}) = v_{t-1}(\mathbf{i}) + \sum_{j=0}^{N-1} \sum_{k=j+1}^{N} \sum_{\ell=1}^{L^{jk}} \lambda^{jk}_{t\ell} \sum_{m=1}^{M^{jk}_\ell} \theta^{jk}_{t\ell m} m K^{jk}_\ell (z^{jk}_{t-1,m}(\mathbf{i}) + c^{jk}_\ell) I(s^{jk}(\mathbf{i}) \geqslant m), \tag{6}$$

where $z^{jk}_{0m}(\mathbf{i}) = 0$ for $s^{jk}(\mathbf{i}) \geqslant m$ and

$$v_1(\mathbf{i}) = \sum_{\ell=1}^{L^{N-1,N}} \lambda^{N-1,N}_{1\ell} \sum_{m=1}^{M^{N-1,N}_\ell} \theta^{N-1,N}_{1\ell m} m K^{N-1,N}_\ell (c^{N-1,N}_\ell) I(s^{N-1,N}(\mathbf{i}) \geqslant m). \tag{7}$$

Before developing the optimal policy, we will prove the following lemma.

LEMMA 2.1.

a. $K^{jk}_\ell(\nu) \geqslant 0$ *for all* ν.

b. 1. If $\nu < \hat{b}^{jk}_\ell$, *then* $K^{jk}_\ell(\nu) > 0$ *and* $\nu < x(\nu) < \hat{b}^{jk}_\ell$.

2. If $\nu = \hat{b}^{jk}_\ell$, *then* $K^{jk}_\ell(\nu) = 0$ *and* $x(\nu) = \hat{b}^{jk}_\ell$.

3. If $\nu > \hat{b}^{jk}_\ell$, *then* $K^{jk}_\ell(\nu) = 0$ *and* $x(\nu) = b^{jk}_\ell$.

c. *1.* $K^{jk}_\ell(\nu)$ *is nonincreasing in* ν.

2. $K^{jk}_\ell(\nu) + \nu$ *is nondecreasing in* ν.

d. $x^{jk}_\ell(\nu)$ *is nondecreasing in* ν.

e. Suppose $\nu_1 \leqslant \nu_2$. *Then* $K^{jk}_\ell(\nu_1) - K^{jk}_\ell(\nu_2) \leqslant \nu_2 - \nu_1$.

f. Let $U = K^{jk}_\ell(\nu_1) - K^{jk}_\ell(\nu_2) + K^{jk}_\ell(\nu_4) - K^{jk}_\ell(\nu_3)$. *Then*

1. if $\nu_1 \leqslant \min\{\nu_2, \nu_3\}$ *and* $\max\{\nu_2, \nu_3\} \leqslant \nu_4$, *then* $U \geqslant p^{jk}_\ell(x^{jk}_\ell(\nu_2))(\nu_2 - \nu_1 + \nu_3 - \nu_4)$.

2. if $\nu_2 \leqslant \min\{\nu_1, \nu_4\}$ *and* $\max\{\nu_1, \nu_4\} \leqslant \nu_3$, *then* $U \leqslant p^{jk}_\ell(x^{jk}_\ell(\nu_1))(\nu_2 - \nu_1 + \nu_3 - \nu_4)$.

Proof. See Appendix. □

From Lemma 2.1b, it follows that the optimal price $x^{jk}_{t\ell m}(\mathbf{i}) = b^{jk}_\ell$ if and only if $z^{jk}_{t-1,m}(\mathbf{i}) + c^{jk}_\ell > \hat{b}^{jk}_\ell$. From these relations, we see that, for any given t and $\mathbf{i}$, a request for booking class ℓ in trip (j, k) should be rejected if and only if $z^{jk}_{t-1,m}(\mathbf{i}) > \hat{b}^{jk}_\ell - c^{jk}_\ell$ (i.e., the booking class ℓ in trip (j, k) should be closed if $z^{jk}_{t-1,m}(\mathbf{i}) > \hat{b}^{jk}_\ell - c^{jk}_\ell$). Note that, if $z^{jk}_{tm}(\mathbf{i})$ is monotonic in i_j, then the rejection rule can be reduced to sets of critical values based on booking capacities, also, if $z^{jk}_{tm}(\mathbf{i})$ is monotonic in t, then the rejection rule can be reduced to sets of critical values based on decision periods,

3. OPTIMAL DECISION POLICIES

3.1 Critical Booking Capacity

In the following subsections, we will show that some sets of critical booking capacities can be obtained in the cases of one or two flight legs without multiple seat bookings.

3.1.1 Single Flight Leg without Multiple Seat Bookings

Here, we will show that a booking policy can be reduced to a set of critical values based on booking capacity for the case of a single flight leg without multiple seat bookings. For simplicity, let $i = i_1$, we shall drop superscript 01 and subscript $m = 1$ from all the related symbols, for example, let $\lambda_{t\ell} = \lambda^{01}_{t\ell}$, $z_t = z^{01}_{t1}$ and so on. Then, from Eqs. 6 and 3 we have

$$v_t(i) = v_{t-1}(i) + \sum_{\ell=1}^{L} \lambda_{t\ell} K_\ell(z_{t-1}(i) - c_\ell) I(i \geqslant 1), \quad t \geqslant 1, \tag{8}$$

$$z_t(i) = z_{t-1}(i) + \sum_{\ell=1}^{L} \lambda_{t\ell}(K_\ell(z_{t-1}(i) + c_\ell) - K_\ell(z_{t-1}(i-1) + c_\ell) I(i \geqslant 2)), \quad t \geqslant 1, \tag{9}$$

where

$$v_1(i) = \sum_{\ell=1}^{L} \lambda_{1\ell} K_\ell(c_\ell) I(i \geq 1), \quad (10)$$

$$z_1(i) = \sum_{\ell=1}^{L} \lambda_{1\ell}(K_\ell(c_\ell) - K_\ell(c_\ell) I(i \geq 2)). \quad (11)$$

THEOREM 3.1 *(a) $z_t(i)$ is nonincreasing in i for all t. (b) $x_{t\ell}(i)$ is nonincreasing in i for all t.*

Proof. See Appendix. □

Theorem 3.1(a) implies that, for a decision period t, there exists a set of critical booking capacities, $\{i(t, \ell)\}$, such that a request for a seat in booking class ℓ is denied if $i < i(t, \ell)$, and satisfied if $i \geq i(t, \ell)$.

REMARK 3.1.

1. Note $x_{t\ell}(i)$ is nonincreasing in i, therefore, if a customer purchases a ticket in a decision period, the price in the next period may increase.
2. The above analysis shows that the booking policy can be reduced to a set of critical booking capacities. This result is similar to that in Lee and Hersh (1993). However, the model we present in this subsection drops the assumption that all arriving customers must purchase tickets. Under this assumption, if we let $c_\ell = 0$ for all ℓ and assume $p_\ell(a_\ell) = 1$ for a price point a_ℓ and set $A_\ell = \{a_\ell, b_\ell\}$ for all ℓ, our model is the same as Lee and Hersh (1993). Then we can see from Lee and Hersh that critical booking capacities for any given t and m may not uniquely exist when multiple seat bookings are allowed.

3.1.2 *Two Flight Legs without Multiple Seat Bookings*

Here, we will show that booking policy can be reduced to sets of critical values based on booking capacity for the case of two flight legs without multiple seat bookings. For simplicity, let us drop the subscript $m = 1$ from all the related symbols. In this case, from Eqs. 6 and 3, we have

$$v_t(\mathbf{i}) = v_{t-1}(\mathbf{i}) + \sum_{j=0}^{1} \sum_{k=j+1}^{2} \sum_{\ell=1}^{L^{jk}} \lambda_{t\ell}^{jk} K_\ell^{jk}(z_{t-1}^{jk}(\mathbf{i}) + c_\ell^{jk}) I(s^{jk}(\mathbf{i}) \geq 1), \quad (12)$$

$$v_1(\mathbf{i}) = \sum_{\ell=1}^{L^{12}} \lambda_{0\ell}^{12} K_\ell^{12}(c_\ell^{12}) I(i_2 \geq 1), \quad (13)$$

where $\mathbf{i} = (i_1, i_2)$. Furthermore, for any integer j^*, k^* such that $s^{j^*k^*}(\mathbf{i}) \geq 1$ and $0 \leq j^* < k^* \leq 2$, we have

$$z_t^{j^*k^*}(\mathbf{i}) = z_{t-1}^{j^*k^*}(\mathbf{i}) + \sum_{j=0}^{1} \sum_{k=j+1}^{2} \sum_{\ell=1}^{L^{jk}} \lambda_{t\ell}^{jk}(K_\ell^{jk}(z_{t-1}^{jk}(\mathbf{i}) + c_\ell^{jk}) I(s^{jk}(\mathbf{i}) \geq 1) - K_\ell^{jk}(z_{t-1}^{jk}(\mathbf{i} - e_{j^*k^*}) + c_\ell^{jk}) I(s^{jk}(\mathbf{i} - e_{j^*k^*}) \geq 1)), \quad (14)$$

$$z_1^{j^*k^*}(\mathbf{i}) = \sum_{\ell=1}^{L^{12}} \lambda_{1\ell}^{12}(K_\ell^{12}(c_\ell^{12}) I(s^{12}(\mathbf{i}) \geq 1) - K_\ell^{12}(c_\ell^{12}) I(s^{12}(\mathbf{i} - e_{j^*k^*}) \geq 1)), \quad (15)$$

where $e_{01} = (1, 0)$, $e_{12} = (0, 1)$, and $e_{02} = (1, 1)$.

THEOREM 3.2.

a. 1. $z_t^{02}(\mathbf{i})$ is nonincreasing in i_1 and i_2 for all t.
2. $z_t^{01}(\mathbf{i})$ is nondecreasing in i_2 for all t.
3. $z_t^{12}(\mathbf{i})$ is nondecreasing in i_1 for all t.

b. 1. $x_{t\ell}^{02}$ are nonincreasing in i_1 and i_2 for all t.
2. $x_{t\ell}^{01}$ is nondecreasing in i_2 for all t.
3. $x_{t\ell}^{12}$ is nondecreasing in i_1 for all t.

Proof. See Appendix. □

The monotonicity of $z_t^{02}(\mathbf{i})$ in i_1 implies that, for a decision period t, booking capacity i_2, in trip (0, 2), there exists a set of critical booking capacities, $\{i_1^{02}(t, i_2, \ell)\}$, such that a request for a seat in booking class ℓ in trip (0, 2) is denied if $i_1 < i_1^{02}(t, i_2, \ell)$, and satisfied if $i_1 \geq i_1^{02}(t, i_2, \ell)$.

The monotonicity of $z_t^{02}(\mathbf{i})$ in i_2 implies that, for a decision period t, booking capacity i_1, in trip (0, 2), there exists a set of critical booking capacities, $\{i_2^{02}(t, i_1, \ell)\}$, such that a request for a seat in booking class ℓ in trip (0, 2) is denied if $i_2 < i_2^{02}(t, i_1, \ell)$, and satisfied if $i_2 \geq i_2^{02}(t, i_1, \ell)$.

Theorem 3.2.a.2. implies that, for a decision period t, booking capacity i_1, in trip (0, 1), there exists a set of critical booking capacities, $\{i_2^{01}(t, i_1, \ell)\}$, such that a request for a seat in booking class ℓ in trip (0, 1) is denied if $i_2 > i_2^{01}(t, i_1, \ell)$, and satisfied if $i_2 \leq i_2^{01}(t, i_1, \ell)$.

Theorem 3.2.a.3. implies that, for a decision period t, booking capacity i_2, in trip (1, 2), there exists a set of critical booking capacities, $\{i_1^{12}(t, i_2, \ell)\}$, such that a request for a seat in booking class ℓ in trip (1, 2) is denied for $i_1 > i_1^{12}(t, i_2, \ell)$, and satisfied for $i_1 \leq i_1^{12}(t, i_2, \ell)$.

TABLE I
Arriving Probabilities for the Numerical Examples

Decision Period	Examples 1 and 2			Example 3								
	λ_{t1}^{01}	λ_{t2}^{01}	λ_{t3}^{01}	λ_{t1}^{01}	λ_{t2}^{01}	λ_{t3}^{01}	λ_{t1}^{12}	λ_{t2}^{12}	λ_{t3}^{12}	λ_{t1}^{02}	λ_{t2}^{02}	λ_{t3}^{02}
1–200	0.08	0.09	0.06	0.00	0.00	0.00	0.07	0.06	0.06	0.00	0.00	0.00
201–400	0.07	0.05	0.07	0.07	0.05	0.07	0.07	0.05	0.04	0.06	0.04	0.07
401–600	0.04	0.03	0.07	0.06	0.02	0.05	0.06	0.04	0.03	0.04	0.03	0.03
601–800	0.06	0.02	0.05	0.03	0.03	0.03	0.05	0.03	0.03	0.03	0.03	0.02
801–1000	0.04	0.03	0.02	0.04	0.03	0.02	0.04	0.03	0.02	0.03	0.02	0.02

THEOREM 3.3.

a. 1. $z_t^{01}(\mathbf{i})$ *is nonincreasing in* i_1 *for all* t.
2. $z_t^{12}(\mathbf{i})$ *is nonincreasing in* i_2 *for all* t.

b. 1. $x_{t,\ell}^{01}$ *is nonincreasing in* i_1 *for all* t.
2. $x_{t,\ell}^{12}$ *is nonincreasing in* i_2 *for all* t.

Proof. See Appendix. □

Theorem 3.3.a.1. implies that, for a decision period t, booking capacity i_2, in trip (0, 1), there exists a set of critical booking capacities, $\{i_1^{01}(t, i_2, \ell)\}$, such that a request for a seat in booking class ℓ in trip (0, 1) is denied for $i_1 < i_1^{01}(t, i_2, \ell)$, and satisfied for $i_1 \geqslant i_1^{01}(t, i_2, \ell)$.

Theorem 3.3.a.2. implies that, for a decision period t, booking capacity i_1, in trip (1, 2), there exists a set of critical booking capacities, $\{i_2^{12}(t, i_1, \ell)\}$, such that a request for a seat in booking class ℓ in trip (1, 2) is denied for $i_2 < i_2^{12}(t, i_1, \ell)$, and satisfied for $i_2 \geqslant i_2^{12}(t, i_1, \ell)$.

3.2 Critical Decision Period

In the previous subsection, we addressed the fact that the critical booking capacity for any given t and m may not uniquely exist when multiple seat bookings are allowed. Thus, the critical booking capacities cannot be applied in controlling the booking process. However, in the following theorem, we will show that the booking process can be reduced to a set of critical decision periods, that also can be used to control the booking process. If the monotonicity of $z_{tm}^{jk}(\mathbf{i})$ in t holds, the assertion turns out to be true.

THEOREM 3.4. (a) $z_{tm}^{jk}(\mathbf{i})$ *is nondecreasing in* t. (b) $x_{t\ell m}^{jk}(\mathbf{i})$ *is nondecreasing in* t.

Proof. See Appendix. □

Theorem 3.4.a implies that, for any given $\mathbf{i}$ such that $s^{jk}(\mathbf{i}) \geqslant m$, in each trip (j, k), there exists a set of critical decision periods, $\{t_{\ell m}^{jk}(\mathbf{i})\}$, such that a request for m seats in booking class ℓ in trip (j, k) is denied for $t > t_{\ell m}^{jk}(\mathbf{i})$, and satisfied for $t \leqslant t_{\ell m}^{jk}(\mathbf{i})$.

REMARK 3.2. Note that $x_{t\ell m}(i)$ is nondecreasing in t, therefore, if no customer arrives in a decision period, the price may decrease in the next decision period.

3.3 Computational Issues

This subsection discusses the issues relating to the reduction in computational requirements.

1. Computational requirement for $K_\ell^{jk}(z_{t-1,m}^{jk}(\mathbf{i}) + c_\ell^{jk})$ can be eliminated when $t > t_\ell^{jk}(\mathbf{i})$.
2. For fixed t such that $t < t_j$, we have $v_t(\mathbf{i}) = v_t(0, \ldots, 0, i_{j+1}, \ldots, i_N)$. This is true because there are no requests for trips including flight leg n, $n = 1, \ldots, j$ when decision period $t < t_j$.
3. For fixed t such that $t > t_j$, and if $i_j \geqslant \max\{M_\ell^{j,j+1}, M_\ell^{j,j+2}, \ldots, M_\ell^{j,N}\}(t - t_j) = y$, then $v_t(\mathbf{i}) = v_t(i_1, \ldots, i_{j-1}, y, i_{j+1}, \ldots, i_N)$. This is true because the number of requests for trips including the flight leg j will be no greater than y.

4. NUMERICAL EXAMPLES

USING THREE EXAMPLES, we will demonstrate some properties of the optimal decision strategy. Examples 1 and 2 are cases of one flight leg, and example 3 is a case of two flight legs. Booking capacities $I_1 = 100$ for examples 1 and 2, whereas $I_1 = 60$ and $I_2 = 70$ for example 3. In all examples, we assume that there are 3 booking classes for all trips. In example 1, the maximum group size is $M_\ell^{jk} = 1$ for all booking classes, whereas $M_\ell^{jk} = 2$ for examples 2 and 3. The total number of decision periods is $T = 1000$ for all examples. All the other related parameters, $\lambda_{t\ell}^{jk}$, $p_\ell^{jk}(x)$, c_ℓ^{jk}, and $\theta_{t\ell m}^{jk}$, are as shown in Tables I to IV.

Example 1 (Single flight leg without multiple seat booking). Figures 2 and 3 show that the optimal price $x_{t\ell}(i)$ is nonincreasing in i for $t = 400$ (Theorem 3.1.b) and nondecreasing in t for $i = 50$ (Theorem 3.4.b), respectively. Figure 2 also shows that, if $0 \leqslant i \leqslant 48$ for booking class 1 and $0 \leqslant i \leqslant 27$ for booking class 2, then the optimal price is a null price, and any subsequently arriving customers in those classes will be always rejected. Similarly, Fig. 2 shows that, if $407 \leqslant t$ for booking class 1 and

TABLE II
Allowable Price Set and Purchasing Probabilities

A_1^{01}	$p_1^{01}(x)$	A_2^{01}	$p_2^{01}(x)$	A_3^{01}	$p_3^{01}(x)$
Examples 1 and 2					
300	0.90	400	0.80	600	0.85
330	0.85	430	0.75	630	0.80
360	0.80	460	0.70	660	0.70
400	0.00	500	0.00	700	0.00
Example 3					
A_1^{01}	$p_1^{01}(x)$	A_2^{01}	$p_2^{01}(x)$	A_3^{01}	$p_3^{01}(x)$
300	0.90	400	0.80	600	0.85
330	0.85	430	0.75	630	0.80
360	0.80	460	0.70	660	0.70
400	0.00	500	0.00	700	0.00
A_1^{12}	$p_1^{12}(x)$	A_1^{12}	$p_2^{12}(x)$	A_1^{12}	$p_3^{12}(x)$
320	0.90	430	0.80	650	0.85
350	0.85	460	0.75	670	0.80
370	0.80	480	0.70	690	0.70
400	0.00	500	0.00	700	0.00
A_1^{02}	$p_1^{02}(x)$	A_1^{02}	$p_2^{02}(x)$	A_1^{02}	$p_3^{02}(x)$
620	0.90	820	0.80	1200	0.85
650	0.85	850	0.75	1220	0.80
670	0.80	870	0.70	1250	0.70
700	0.00	900	0.00	1300	0.00

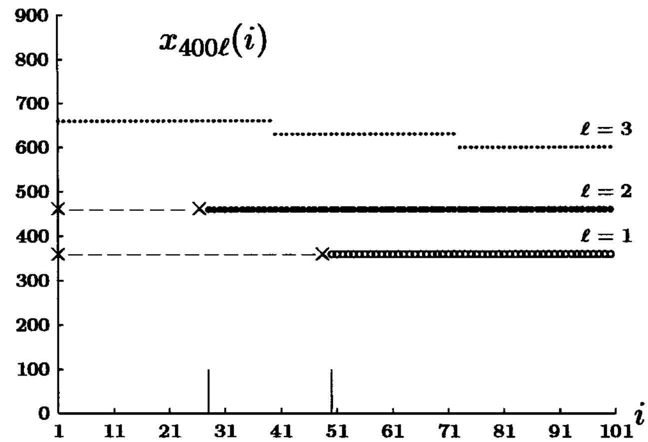

Fig. 2. Optimal price when $t = 400$.

$870 \leqslant t$ for booking class 2, then the optimal price is a null price, so any subsequently arriving customers shall be rejected. Figures 4 and 5 show that $t_\ell(i)$ is nondecreasing in i and $i(t, \ell)$ is nondecreasing in t, respectively, where $t_3(i) = 1000$ for all i and $i(t, 3) = 1$ for all t. These relationships imply that any arriving customer of class $\ell = 3$ should be satisfied. Figure 4 can be interpreted as follows: if, for example $i = 50$, then a customer of each of classes 1, 2, and 3 should be satisfied if and only if $t \leqslant 406$, $t \leqslant 869$, and $t \leqslant 1000$, respectively. Similarly, Fig. 5 tells us that, for example, if $t = 400$, then a customer of class 1, 2, and 3 should be satisfied if and only if $i \geqslant 50$, $i \geqslant 28$ and $i \geqslant 1$, respectively.

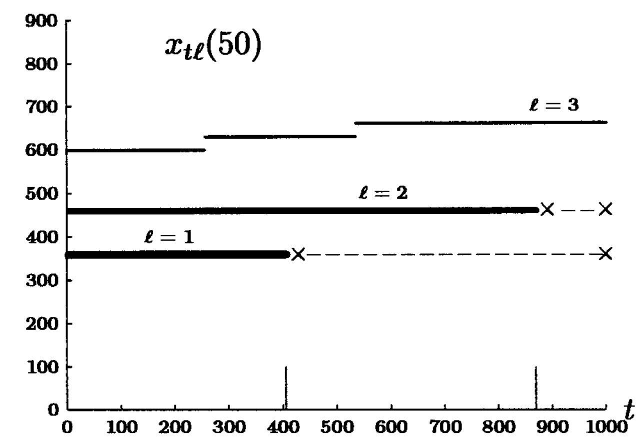

Fig. 3. Optimal price when $i = 50$.

TABLE III
Fixed Cost Incurred from Carrying Passengers

c_1^{01}	c_2^{01}	c_3^{01}	c_1^{12}	c_2^{12}	c_3^{12}	c_1^{02}	c_2^{02}	c_3^{02}
20	40	50	15	35	45	30	70	90

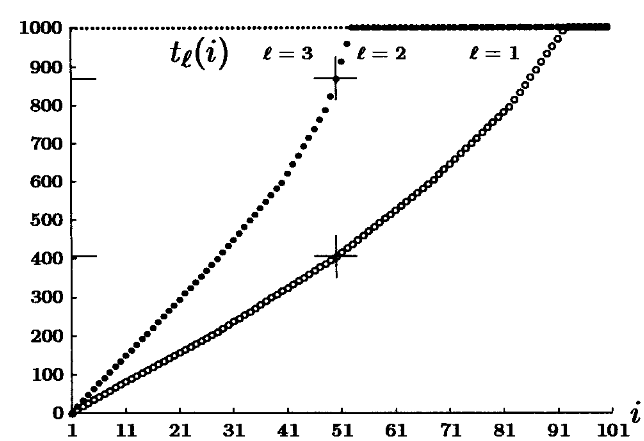

Fig. 4. Critical decision period.

TABLE IV
Probabilities for Multiple Seat Bookings

Trip	Decision Period	θ_{t1m}^{jk} $m = 1$	θ_{t1m}^{jk} $m = 2$	θ_{t2m}^{jk} $m = 1$	θ_{t2m}^{jk} $m = 2$	θ_{t3m}^{jk} $m = 1$	θ_{t3m}^{jk} $m = 2$
Example 2							
(0, 1)	1–1000	0.20	0.80	0.25	0.75	0.25	0.75
Example 3 ($t_1 = 201$)							
(0, 1)	201–1000	0.20	0.80	0.25	0.75	0.25	0.75
(1, 2)	1–800	0.20	0.80	0.25	0.75	0.50	0.50
(1, 2)	801–1000	0.50	0.50	0.50	0.50	0.50	0.50
(0, 2)	201–800	0.20	0.80	0.25	0.75	0.50	0.50
(0, 2)	801–1000	0.50	0.50	0.50	0.50	0.50	0.50

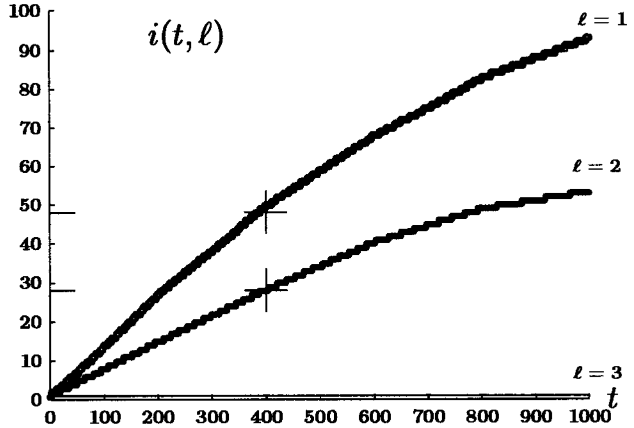

Fig. 5. Critical booking capacity.

Example 2 (Single flight leg with maximum group size $M_\ell = 2$). Figure 6 shows the optimal price $x_{t\ell m}(i)$ of $t = 20$ and $m = 1$ for each of classes 1, 2, and 3. It becomes almost monotonic in i as in example 1; however, as seen in Fig. 7 (which is a magnified figure of the neighborhood of $i = 1$ in Fig. 6), it is not monotonic in i for class 3 and is reduced to a null price at $i = 2$ and 4 for class 1 and 2. The nonmonotonicity of $x_{t\ell m}(i)$ and the appearance of the null price come from the nonmonotonicity of $t_{\ell m}(i)$ in i as has been illustrated in Fig. 9, which is a magnified figure of the neighborhood of $i = 1$ in Fig. 8.

Example 3 (Two flight legs with maximum group size $M_\ell^{jk} = 2$). The interpretation of Figs. 10 through 13 is almost the same as Figs. 2 through 5.

5. FUTURE WORK

THIS PAPER PRESENTED a general model with certain restrictive assumptions. To evolve into a more realistic model, it will be necessary to investigate cases such as no-show, overbooking, and so on. Moreover, because the computational time for solving this model increases with the number of flight legs and with the maximum booking capacity in each flight leg, it will be necessary to develop techniques for achieving further reduction in computational time as the flight leg size increases.

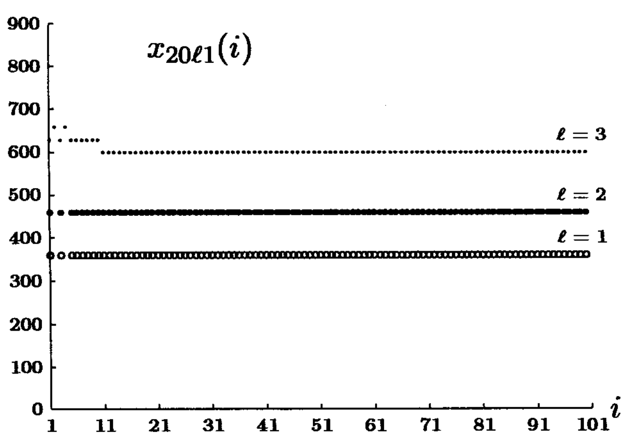

Fig. 6. Optimal price when $t = 20$ and $M_\ell = 1$.

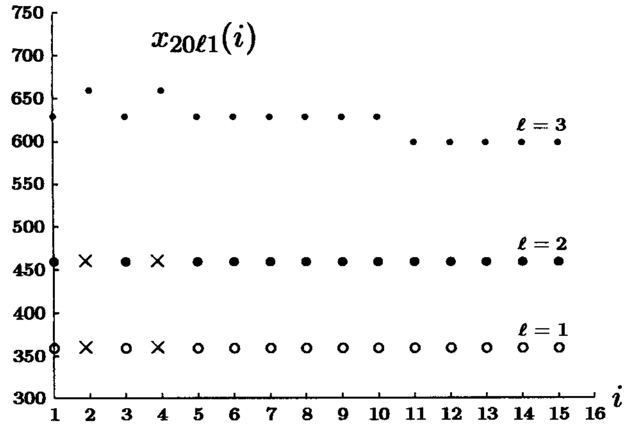

Fig. 7. A magnified figure of the neighborhood at ℓ_1 in Fig. 6.

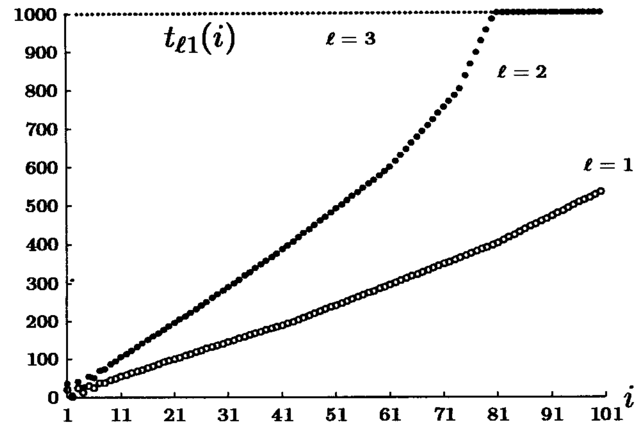

Fig. 8. Critical decision period.

APPENDIX

Proof of Lemma 2.1. For simplicity of the expressions, we shall drop superscript jk and subscript ℓ from all the related symbols throughout the proof.

(a) It is clear that $K(\nu) \geqslant p(b)(b - \nu) = 0$ for all ν.
(b1) $p(x)(x - \nu) \leqslant 0$ for $x \leqslant \nu$ always, $p(x)(x - \nu) > 0$ for $\nu < x \leqslant \hat{b}$, and $p(x)(x - \nu) = 0$ for $x = \hat{b}$, hence $K(\nu) > 0$ and $\nu < x(\nu) \leqslant \hat{b}$.
(b2) If $\nu = \hat{b}$, then $p(x)(x - \nu) < 0$ for $x < \hat{b}$ and $p(x)(x - \nu) = 0$ for $x \geqslant \hat{b}$, hence $K(\nu) = 0$ and $x(\nu) = \hat{b}$ from the definition of $x(\nu)$.
(b3) If $\nu > \hat{b}$, then $p(x)(x - \nu) < 0$ for $x \leqslant \hat{b}$ and

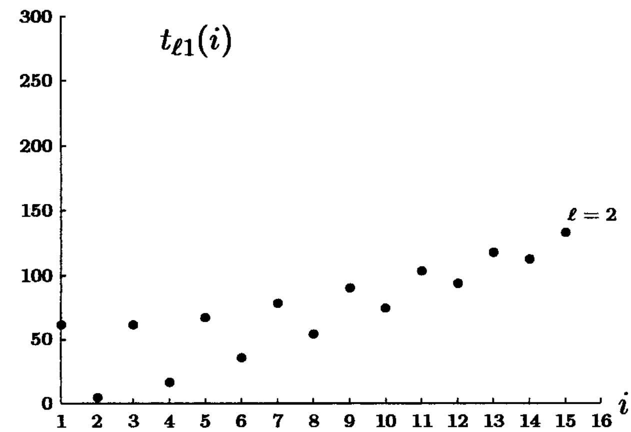

Fig. 9. A magnified figure of the neighborhood of $\ell_1 = 1$ in Fig. 8.

$p(x)(x - \nu) = 0$ for $x = b$, hence $K(\nu) = 0$ and $x(\nu) = b$ from the definition of $x(\nu)$.

(c1) It is obvious that $p(x)(x - \nu)$ is nonincreasing in ν for all x.

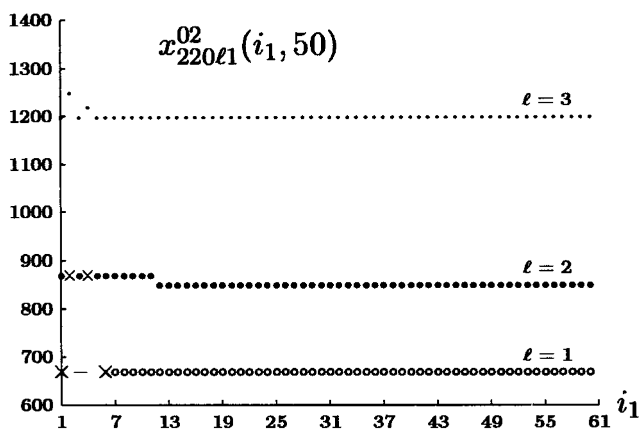

Fig. 10. Optimal price for (0, 2) leg when $M_\ell = 1$, $t = 220$, and $i_2 = 50$.

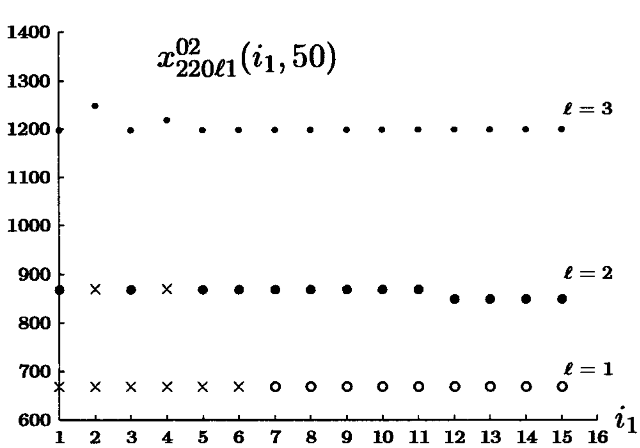

Fig. 11. A magnified figure of the neighborhood of $i_1 = 1$ in Fig. 10.

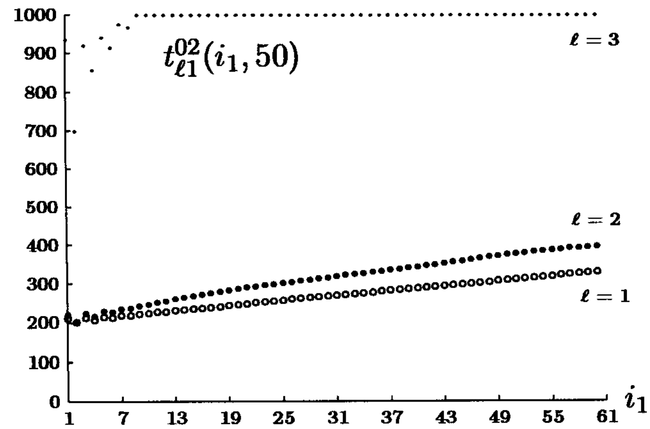

Fig. 12. Critical decision period is not monotonic in t.

(c2) Clearly, $p(x)(x - \nu) + \nu$ $(= p(x)x + (1 - p(x))\nu)$ is nondecreasing in ν for all x.

(d) For any $\xi > 0$ we have

$$\begin{aligned} K(\nu + \xi) &= \max_{x \in A} p(x)(x - (\nu + \xi)) \\ &= p(x(\nu + \xi))(x(\nu + \xi) - (\nu + \xi)) \\ &= p(x(\nu + \xi))(x(\nu + \xi) - \nu) \\ &\quad - p(x(\nu + \xi))\xi. \end{aligned}$$

Because $p(x(\nu + \xi))(x(\nu + \xi) - \nu) \leqslant K(\nu) = p(x(\nu))(x(\nu) - \nu)$, we have

$$\begin{aligned} K(\nu + \xi) &\leqslant p(x(\nu))(x(\nu) - \nu) - p(x(\nu + \xi))\xi \\ &= p(x(\nu))(x(\nu) - (\nu + \xi)) \\ &\quad + \xi(p(x(\nu)) - p(x(\nu + \xi))). \end{aligned}$$

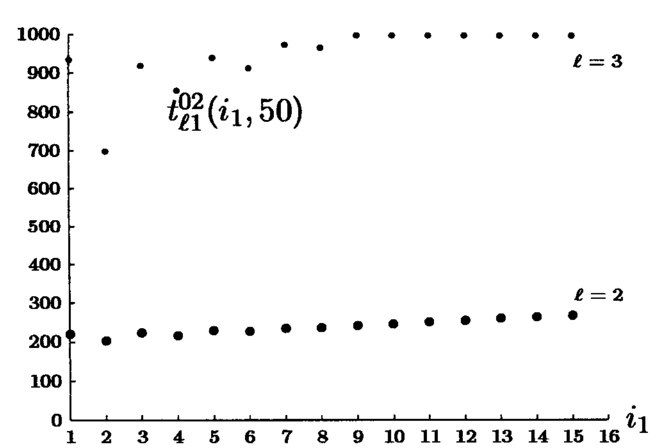

Fig. 13. A magnified figure of the neighborhood of $i_1 = 1$ in Fig. 12.

Because $p(x(\nu))(x(\nu) - (\nu + \xi)) \leqslant K(\nu + \xi)$, we can show that

$$K(\nu + \xi) \leqslant K(\nu + \xi) + \xi(p(x(\nu)) - p(x(\nu + \xi))),$$

from which we have $0 \leqslant p(x(\nu)) - p(x(\nu + \xi))$, that is, $p(x(\nu)) \geqslant p(x(\nu + \xi))$, therefore $x(\nu) \leqslant x(\nu + \xi)$ because $p(x)$ is strictly decreasing for $x \leqslant b$.

(e) Suppose $\nu_1 \leqslant \nu_2$. Then, because $K(\nu_1) + \nu_1 \leqslant K(\nu_2) + \nu_2$ (from (c2)), it is clear that $K(\nu_1) - K(\nu_2) \leqslant \nu_2 - \nu_1$.

(f1) Note that, for any real numbers a and b, we have

$$\begin{aligned} K(a) - K(b) &= \max_{x\in A} p(x)(x - a) - \max_{x\in A} p(x)(x - b) \\ &\geqslant p(x(b))(x(b) - a) - p(x(b))(x(b) - b) \\ &= p(x(b))(b - a). \end{aligned}$$

Now let us analyze the following two possible cases:

1. Case of $\nu_2 \leqslant \nu_3$: In this case, immediately we have $p(x(\nu_3)) \leqslant p(x(\nu_2))$ due to $x(\nu_2) \leqslant x(\nu_3)$ from (d), therefore, we have

$$\begin{aligned} U &= K(\nu_1) - K(\nu_2) + K(\nu_4) - K(\nu_3) \\ &\geqslant p(x(\nu_2))(\nu_2 - \nu_1) + p(x(\nu_3))(\nu_3 - \nu_4) \\ &\geqslant p(x(\nu_2))(\nu_2 - \nu_1) + p(x(\nu_2))(\nu_3 - \nu_4) \\ &= p(x(\nu_2))(\nu_2 - \nu_1 + \nu_3 - \nu_4). \end{aligned}$$

2. Case of $\nu_2 \geqslant \nu_3$: In the same way as above, we can show that $p(x(\nu_2)) \leqslant p(x(\nu_3))$, therefore, we have

$$\begin{aligned} U &= K(\nu_1) - K(\nu_3) + K(\nu_4) - K(\nu_2) \\ &\geqslant p(x(\nu_3))(\nu_3 - \nu_1) + p(x(\nu_2))(\nu_2 - \nu_4) \\ &\geqslant p(x(\nu_2))(\nu_3 - \nu_1) + p(x(\nu_2))(\nu_2 - \nu_4) \\ &= p(x(\nu_2))(\nu_2 - \nu_1 + \nu_3 - \nu_4). \end{aligned}$$

(f2) Note that, for any real numbers a and b, we have

$$\begin{aligned} K(a) - K(b) &= \max_{x\in A} p(x)(x - a) - \max_{x\in A} p(x)(x - b) \\ &\leqslant p(x(a))(b - a). \end{aligned}$$

Now, let us analyze the following two possible cases:

1. Case of $\nu_1 \leqslant \nu_4$: In the same way as (f1), we can show that $p(x(\nu_4)) \leqslant p(x(\nu_1))$, therefore, we have

$$\begin{aligned} U &= K(\nu_1) - K(\nu_2) + K(\nu_4) - K(\nu_3) \\ &\leqslant p(x(\nu_1))(\nu_2 - \nu_1) + p(x(\nu_4))(\nu_3 - \nu_4) \\ &\leqslant p(x(\nu_1))(\nu_2 - \nu_1) + p(x(\nu_1))(\nu_3 - \nu_4) \\ &= p(x(\nu_1))(\nu_2 - \nu_1 + \nu_3 - \nu_4). \end{aligned}$$

2. Case of $\nu_1 \geqslant \nu_4$: In the same way as (f1), we can show that $p(x(\nu_1)) \leqslant p(x(\nu_4))$, therefore we have

$$\begin{aligned} U &= K(\nu_1) - K(\nu_3) + K(\nu_4) + K(\nu_2) \\ &\leqslant p(x(\nu_1))(\nu_3 - \nu_1) + p(x(\nu_4))(\nu_2 - \nu_4) \\ &\leqslant p(x(\nu_1))(\nu_3 - \nu_1) + p(x(\nu_1))(\nu_2 - \nu_4) \\ &= p(x(\nu_1))(\nu_2 - \nu_1 + \nu_3 - \nu_4). \qquad \square \end{aligned}$$

Proof of Theorem 3.1. (a) First, it can be easily shown that

$$z_1(2) - z_1(1) = \sum_{\ell=1}^{L} \lambda_{1\ell} K_\ell(c_\ell) \leqslant 0,$$

and $z_1(i) - z_1(i - 1) = 0$ for $i \geqslant 3$. Hence, the assertion holds for $t = 1$. Next, assume that the assertion holds for some $t \geqslant 2$. Then, from Eq. 9, we have for $i \geqslant 2$,

$$\begin{aligned} z_t(i) - z_t(i-1) &= z_{t-1}(i) - z_{t-1}(i-1) \\ &\quad + \sum_{\ell=1}^{L} \lambda_{t\ell}(K_\ell(z_{t-1}(i) + c_\ell) - K_\ell(z_{t-1}(i-1) + c_\ell) \\ &\quad + K_\ell(z_{t-1}(i-2) + c_\ell)I(i \geqslant 3) \\ &\quad - K_\ell(z_{t-1}(i-1) + c_\ell)). \end{aligned}$$

Because $K_\ell(z_{t-1}(i-2) + c_\ell)I(i \geqslant 3) - K_\ell(z_{t-1}(i-1) + c_\ell) = -K_\ell(z_{t-1}(i-1) + c_\ell) \leqslant 0$ for $i = 2$ from Lemma 2.1(a), $K_\ell(z_{t-1}(i-2) + c_\ell)I(i \geqslant 3) - K_\ell(z_{t-1}(i-1) + c_\ell) \leqslant 0$ for $i \geqslant 3$ from Lemma 2.1(c1), and $K_\ell(z_{t-1}(i) + c_\ell) - K_\ell(z_{t-1}(i-1) + c_\ell) \leqslant z_{t-1}(i-1) - z_{t-1}(i)$ for $i \geqslant 2$ from Lemma 2.1(e), we have

$$\begin{aligned} z_t(i) - z_t(i-1) &\leqslant z_{t-1}(i-1) - z_{t-1}(i) \\ &\quad + \sum_{\ell=1}^{L} \lambda_{t\ell}(z_{t-1}(i-1) - z_{t-1}(i)) \\ &= \left(1 - \sum_{\ell=1}^{L} \lambda_{t\ell}\right)(z_{t-1}(i) - z_{t-1}(i-1)) \leqslant 0. \end{aligned}$$

(b) Immediately from (a) and Lemma 2.1(d). □

Proof of Theorem 3.2. (a) Note that

$$\begin{aligned} z_t^{12}(\mathbf{i}) - z_t^{12}(\mathbf{i} - e_{01}) &= v_t(\mathbf{i}) - v_t(\mathbf{i} - e_{12}) - v_t(\mathbf{i} - e_{01}) + v_t(\mathbf{i} - e_{02}) \\ &= v_t(\mathbf{i}) - v_t(\mathbf{i} - e_{01}) - v_t(\mathbf{i} - e_{12}) + v_t(\mathbf{i} - e_{02}) \\ &= z_t^{01}(\mathbf{i}) - z_t^{01}(\mathbf{i} - e_{12}). \end{aligned}$$

Similarly, we have

$$z_t^{01}(\mathbf{i}) - z_t^{01}(\mathbf{i} - e_{02}) = z_t^{02}(\mathbf{i}) - z_t^{02}(\mathbf{i} - e_{01}),$$

$$z_t^{12}(\mathbf{i}) - z_t^{12}(\mathbf{i} - e_{02}) = z_t^{02}(\mathbf{i}) - z_t^{02}(\mathbf{i} - e_{12}).$$

Hence, if (a3) holds, then so also does (a2), and if (a1) holds, then so also does

$$z_t^{01}(\mathbf{i}) \leqslant z_t^{01}(\mathbf{i} - e_{02}), \quad i_1 \geqslant 2, \quad i_2 \geqslant 1, \quad t \geqslant 0, \tag{16}$$

$$z_{t-1}^{12}(\mathbf{i}) \leqslant z_{t-1}^{12}(\mathbf{i} - e_{02}), \quad i_1 \geqslant 1, \quad i_2 \geqslant 2, \quad t \geqslant 0. \tag{17}$$

Now, let us prove all assertions simultaneously by induction. First, for $t = 1$, from Eq. 15 we have

$$z_1^{02}(\mathbf{i}) - z_1^{02}(\mathbf{i} - e_{01}) = 0, \quad \text{for} \quad i_1 \geqslant 2, i_2 \geqslant 1,$$

$$z_1^{02}(\mathbf{i}) - z_1^{02}(\mathbf{i} - e_{12}) \leqslant 0, \quad \text{for} \quad i_1 \geqslant 1, i_2 \geqslant 2,$$

$$z_1^{12}(\mathbf{i}) - z_1^{12}(\mathbf{i} - e_{01}) = 0, \quad \text{for} \quad i_1 \geqslant 1, i_2 \geqslant 1.$$

Therefore, the assertions (a1) and (a3) hold for $t = 1$. Assume that they hold for some $t \geqslant 2$. Then, $z_{t-1}^{02}(\mathbf{i})$ is nonincreasing in i_1 and i_2, and $z_{t-1}^{01}(\mathbf{i})$ is nondecreasing in i_2, so $z_t^{12}(\mathbf{i})$ is nondecreasing in i_1, $z_{t-1}^{01}(\mathbf{i}) \leqslant z_{t-1}^{01}(\mathbf{i} - e_{02})$ and $z_{t-1}^{12}(\mathbf{i}) \leqslant z_{t-1}^{12}(\mathbf{i} - e_{02})$. From Eq. 14, we have, for $i_1 \geqslant 1$ and $i_2 \geqslant 1$,

$$z_t^{12}(\mathbf{i}) - z_t^{12}(\mathbf{i} - e_{01}) = G_1 + \sum_{\ell=1}^{L^{01}} \lambda_{t\ell}^{01} G_2 + \sum_{\ell=1}^{L^{12}} \lambda_{t\ell}^{12} G_3 + \sum_{\ell=1}^{L^{02}} \lambda_{t\ell}^{02} G_4 \tag{18}$$

where

$$G_1 = z_{t-1}^{12}(\mathbf{i}) - z_{t-1}^{12}(\mathbf{i} - e_{01}),$$

$$G_2 = G_2^1 - G_2^2 - G_2^3 + G_2^4,$$

$$G_2^1 = K_\ell^{01}(z_{t-1}^{01}(\mathbf{i}) + c_\ell^{01}),$$

$$G_2^2 = K_\ell^{01}(z_{t-1}^{01}(\mathbf{i} - e_{12}) + c_\ell^{01}),$$

$$G_2^3 = K_\ell^{01}(z_{t-1}^{01}(\mathbf{i} - e_{01}) + c_\ell^{01}) I(i_1 \geqslant 2),$$

$$G_2^4 = K_\ell^{01}(z_{t-1}^{01}(\mathbf{i} - e_{02}) + c_\ell^{01}) I(i_1 \geqslant 2),$$

$$G_3 = G_3^1 - G_3^2 - G_3^3 + G_3^4,$$

$$G_3^1 = K_\ell^{12}(z_{t-1}^{12}(\mathbf{i}) + c_\ell^{12}),$$

$$G_3^2 = K_\ell^{12}(z_{t-1}^{12}(\mathbf{i} - e_{12}) + c_\ell^{12}) I(i_2 \geqslant 2),$$

$$G_3^3 = K_\ell^{12}(z_{t-1}^{12}(\mathbf{i} - e_{01}) + c_\ell^{12}),$$

$$G_3^4 = K_\ell^{12}(z_{t-1}^{12}(\mathbf{i} - e_{02}) + c_\ell^{12}) I(i_2 \geqslant 2),$$

$$G_4 = G_4^1 - G_4^2 - G_4^3 + G_4^4,$$

$$G_4^1 = K_\ell^{02}(z_{t-1}^{02}(\mathbf{i}) + c_\ell^{02}),$$

$$G_4^2 = K_\ell^{02}(z_{t-1}^{02}(\mathbf{i} - e_{12}) + c_\ell^{02}) I(i_2 \geqslant 2),$$

$$G_4^3 = K_\ell^{02}(z_{t-1}^{02}(\mathbf{i} - e_{01}) + c_\ell^{02}) I(i_1 \geqslant 2),$$

$$G_4^4 = K_\ell^{02}(z_{t-1}^{02}(\mathbf{i} - e_{02}) + c_\ell^{02}) I(i_1, i_2 \geqslant 2).$$

Clearly, $G_1 \geqslant 0$ from the induction hypothesis. Note that, if we can show that $G_j \geqslant -G_1$ for $j = 2$, 3, and 4, then the result is clear from Eq. 18. For G_2, because $z_{t-1}^{01}(\mathbf{i})$ is nondecreasing in i_2 from the induction hypothesis, we have $-G_2^3 + G_2^4 \geqslant 0$ from Lemma 2.1(c1) and

$$\begin{aligned} G_2^1 - G_2^2 &\geqslant z_{t-1}^{01}(\mathbf{i} - e_{12}) - z_{t-1}^{01}(\mathbf{i}) \\ &= z_{t-1}^{12}(\mathbf{i} - e_{01}) - z_{t-1}^{12}(\mathbf{i}) \\ &= -G_1 \end{aligned}$$

from Lemma 2.1(e). Therefore,

$$G_2 \geqslant -G_1. \tag{19}$$

For G_3, because $z_{t-1}^{12}(\mathbf{i})$ is nondecreasing in i_1 from the induction hypothesis, we have $-G_3^2 + G_3^4 \geqslant 0$ from Lemma 2.1(c1) and $G_3^1 - G_3^3 \geqslant z_{t-1}^{12}(\mathbf{i} - e_{01}) - z_{t-1}^{12}(\mathbf{i}) = -G_1$ from Lemma 2.1(e). Therefore,

$$G_3 \geqslant -G_1. \tag{20}$$

For G_4, it will suffice to verify that $G_4 \geqslant -G_1$ for the following two cases:

Case $I(i_1, i_2 \geqslant 2) = 0$. Because $z_{t-1}^{02}(\mathbf{i})$ is nonincreasing in i_1 from the induction hypothesis, by Lemma 2.1(c1) we have $G_4 = G_4^1 - G_4^2 \geqslant 0$ for $i_1 = 1$, $i_2 \geqslant 2$ and $G_4 = G_4^1 - G_4^3 \geqslant 0$ for $i_1 \geqslant 2$, $i_2 = 1$.

Case $I(i_1, i_2 \geqslant 2) = 1$. By the induction hypothesis and inequality 16, we have $z_{t-1}^{02}(\mathbf{i}) \leqslant \min\{z_{t-1}^{02}(\mathbf{i} - e_{12}), z_{t-1}^{02}(\mathbf{i} - e_{01})\}$ and $\max\{z_{t-1}^{02}(\mathbf{i} - e_{12}), z_{t-1}^{02}(\mathbf{i} - e_{01})\} \leqslant z_{t-1}^{02}(\mathbf{i} - e_{02})$. Therefore, by Lemma 2.1(f1), we have

$$\begin{aligned} G_4 \geqslant\; & p_\ell^{02}(x_t^{02}(\mathbf{i} - e_{12}))(z_{t-1}^{02}(\mathbf{i} - e_{12}) \\ & - z_{t-1}^{02}(\mathbf{i}) - z_{t-1}^{02}(\mathbf{i} - e_{02}) + z_{t-1}^{02}(\mathbf{i} - e_{01})) \\ =\; & p_\ell^{02}(x_{t\ell 1}^{02}(\mathbf{i} - e_{12}))(z_{t-1}^{12}(\mathbf{i} - e_{01}) \end{aligned}$$

$$- z^{12}_{t-1}(\mathbf{i}) + z^{12}_{t-1}(\mathbf{i} - e_{02}) - z^{12}_{t-1}(\mathbf{i} - e_{02} - e_{01}))$$

$$\geqslant p^{02}_{\ell}(x^{02}_{t\ell 1}(\mathbf{i} - e_{12}))(z^{12}_{t-1}(\mathbf{i} - e_{01}) - z^{12}_{t-1}(\mathbf{i}))$$

$$\geqslant z^{12}_{t-1}(\mathbf{i} - e_{01}) - z^{12}_{t-1}(\mathbf{i}) = -G_1.$$

where the second inequality follows from induction that $z^{12}_{t-1}(\mathbf{i})$ is nondecreasing in i_1. Therefore,

$$G_4 \geqslant -G_1. \tag{21}$$

Substituting inequalities 19, 20, and 21 into Eq. 18, we get

$$z^{12}_t(\mathbf{i}) - z^{12}_t(\mathbf{i} - e_{01})$$

$$\geqslant \left(1 - \sum_{\ell=1}^{L^{01}} \lambda^{01}_{t\ell} - \sum_{\ell=1}^{L^{12}} \lambda^{12}_{t\ell} - \sum_{\ell=1}^{L^{02}} \lambda^{02}_{t\ell}\right) G_1 \geqslant 0.$$

Hence, the assertion (a3), hence (a2) holds. Next, let us show the proof of the monotonicity $z^{02}_t(\mathbf{i})$ in i_1 for t. From Eq. 14, we have, for $i_1 \geqslant 2$ and $i_2 \geqslant 1$,

$$z^{02}_t(\mathbf{i}) - z^{02}_t(\mathbf{i} - e_{01})$$

$$= Q_1 + \sum_{\ell=1}^{L01} \lambda^{01}_{t\ell} Q_2 + \sum_{\ell=1}^{L^{12}} \lambda^{12}_{t\ell} Q_3 + \sum_{\ell=1}^{L^{02}} \lambda^{02}_{t\ell} Q_4 \tag{22}$$

where

$$Q_1 = z^{02}_{t-1}(\mathbf{i}) - z^{02}_{t-1}(\mathbf{i} - e_{01}),$$

$$Q_2 = Q^1_2 - Q^2_2 - Q^3_2 + Q^4_2,$$

$$Q^1_2 = K^{01}_{\ell}(z^{01}_{t-1}(\mathbf{i}) + c^{01}_{\ell}),$$

$$Q^2_2 = K^{01}_{\ell}(z^{01}_{t-1}(\mathbf{i} - e_{02}) + c^{01}_{\ell}),$$

$$Q^3_2 = K^{01}_{\ell}(z^{01}_{t-1}(\mathbf{i} - e_{01}) + c^{01}_{\ell}),$$

$$Q^4_2 = K^{01}_{\ell}(z^{01}_{t-1}(\mathbf{i} - e_{02} - e_{01}) + c^{01}_{\ell}) I(i_1 \geqslant 3),$$

$$Q_3 = Q^1_3 - Q^2_3 - Q^3_3 + Q^4_3,$$

$$Q^1_3 = K^{12}_{\ell}(z^{12}_{t-1}(\mathbf{i}) + c^{12}_{\ell}),$$

$$Q^2_3 = K^{12}_{\ell}(z^{12}_{t-1}(\mathbf{i} - e_{02}) + c^{12}_{\ell}) I(i_2 \geqslant 2),$$

$$Q^3_3 = K^{12}_{\ell}(z^{12}_{t-1}(\mathbf{i} - e_{01}) + c^{12}_{\ell}),$$

$$Q^4_3 = K^{12}_{\ell}(z^{12}_{t-1}(\mathbf{i} - e_{02} - e_{01}) + c^{12}_{\ell}) I(i_2 \geqslant 2),$$

$$Q_4 = Q^1_4 - Q^2_4 - Q^3_4 + Q^4_4,$$

$$Q^1_4 = K^{02}_{\ell}(z^{02}_{t-1}(\mathbf{i}) + c^{02}_{\ell}),$$

$$Q^2_4 = K^{02}_{\ell}(z^{02}_{t-1}(\mathbf{i} - e_{02}) + c^{02}_{\ell}) I(i_2 \geqslant 2),$$

$$Q^3_4 = K^{02}_{\ell}(z^{02}_{t-1}(\mathbf{i} - e_{01}) + c^{02}_{\ell}),$$

$$Q^4_4 = K^{02}_{\ell}(z^{02}_{t-1}(\mathbf{i} - e_{02} - e_{01}) + c^{02}_{\ell}) I(i_1 \geqslant 3,\ i_2 \geqslant 2).$$

Clearly, we have $Q_1 \leqslant 0$ from the induction hypothesis. For Q_2, because $z^{01}_{t-1}(\mathbf{i}) \leqslant z^{01}_{t-1}(\mathbf{i} - e_{02})$ from the induction hypothesis, we have $-Q^3_2 + Q^4_2 = -Q^3_2 \leqslant 0$ for $i_1 < 3$ from Lemma 2.1(a), $-Q^3_2 + Q^4_2 \leqslant 0$ for $i \geqslant 3$ from Lemma 2.1(c1) and $Q^1_2 - Q^2_2 \leqslant z^{01}_{t-1}(\mathbf{i} - e_{02}) - z^{01}_{t-1}(\mathbf{i}) = z^{02}_{t-1}(\mathbf{i}) - z^{02}_{t-1}(\mathbf{i} - e_{01}) = -Q_1$ from Lemma 2.1(e). Therefore,

$$Q_2 \leqslant -Q_1. \tag{23}$$

For Q_3, it will suffice to verify that $Q_3 \leqslant -Q_1$ in the following two cases:

Case $I(i_2 \geqslant 2) = 0$. In this case, we have $Q_3 = Q^1_3 - Q^3_3$. Because $z^{12}_{t-1}(\mathbf{i}) \geqslant z^{12}_{t-1}(\mathbf{i} - e_{01})$ from the induction hypothesis, we get $Q^1_3 - Q^3_3 \leqslant 0$ from Lemma 2.1(c1).

Case $I(i_2 \geqslant 2) = 1$. By the induction hypothesis and inequality 17, we have $z^{12}_{t-1}(\mathbf{i} - e_{01}) \leqslant \min\{z^{12}_{t-1}(\mathbf{i}),\ z^{12}_{t-1}(\mathbf{i} - e_{02} - e_{01})\}$ and $\max\{z^{12}_{t-1}(\mathbf{i}),\ z^{12}_{t-1}(\mathbf{i} - e_{02} - e_{01})\} \leqslant z^{12}_{t-1}(\mathbf{i} - e_{02})$. Hence, from Lemma 2.1(f2), we have

$$Q_3 \leqslant p^{12}_{\ell}(x^{12}_{t\ell,1}(\mathbf{i}))(z^{12}_{t-1}(\mathbf{i} - e_{01}) \leftarrow z^{12}_{t-1}(\mathbf{i})$$

$$+ z^{12}_{t-1}(\mathbf{i} - e_{02}) - z^{12}_{t-1}(\mathbf{i} - e_{02} - e_{01}))$$

$$= p^{12}_{\ell}(x^{12}_{t\ell,1}(\mathbf{i}))(z^{02}_{t-1}(\mathbf{i} - e_{01}) - z^{02}_{t-1}(\mathbf{i})$$

$$+ z^{02}_{t-1}(\mathbf{i} - e_{12}) - z^{02}_{t-1}(\mathbf{i} - e_{02}))$$

$$\leqslant p^{12}_{\ell}(x^{12}_{t\ell,1}(\mathbf{i}))(z^{02}_{t-1}(\mathbf{i} - e_{01}) - z^{02}_{t-1}(\mathbf{i}))$$

$$\leqslant z^{02}_{t-1}(\mathbf{i} - e_{01}) - z^{02}_{t-1}(\mathbf{i}) = -Q_1,$$

where the second inequality follows from the induction that $z^{02}_{t-1}(\mathbf{i})$ is nonincreasing in i_1. Therefore,

$$Q_3 \leqslant -Q_1. \tag{24}$$

For Q_4, because $-Q^2_4 + Q^4_4 = 0$ for $i_2 = 1$, $-Q^2_4 + Q^4_4 = -Q^2_4 \leqslant 0$ for $i_1 = 2$ and $i_2 \geqslant 2$ from Lemma 2.1(a), $-Q^2_4 + Q^4_4 \leqslant 0$ for $i_1 \geqslant 3$ and $i_2 \geqslant 2$ from Lemma 2.1(c1), we get $-Q^2_4 + Q^4_4 \leqslant 0$, and because $Q^1_4 - Q^3_4 \leqslant z^{02}_{t-1}(\mathbf{i} - e_{01}) - z^{02}_{t-1}(\mathbf{i}) = -Q_1$ from Lemma 2.1(e). Therefore,

$$Q_4 \leqslant -Q_1. \tag{25}$$

Substituting inequalities 23, 24, and 25 into Eq. 22, we get

$$z^{02}_t(\mathbf{i}) - z^{02}_t(\mathbf{i} - e_{01})$$

$$\leqslant \left(1 - \sum_{\ell=1}^{L01} \lambda^{01}_{t\ell} - \sum_{\ell=1}^{L^{12}} \lambda^{12}_{t\ell} - \sum_{\ell=1}^{L^{02}} \lambda^{02}_{t\ell}\right) Q_1 \leqslant 0.$$

Hence, it follows that $z^{02}_t(\mathbf{i})$ is nonincreasing in i_1. In the same way, we can show that $z^{02}_t(\mathbf{i}) - z^{02}_t(\mathbf{i} - e_{12}) \leqslant 0$. We have completed the proof.

(b) The monotonicity of $x^{01}_{t\ell}(\mathbf{i})$ in i_1, $x^{12}_{t\ell}(\mathbf{i})$ in i_2 and

$x^{02}_{t\ell}(\mathbf{i})$ in i_1 and i_2 are immediate results from (a) and Lemma 2.1(d). □

Proof of Theorem 3.3. (a) From the definition of $z^{02}_t(\mathbf{i})$, we have

$$z^{02}_t(\mathbf{i}) = z^{01}_t(\mathbf{i}) + z^2_t(\mathbf{i} - e_{01}).$$

Because $z^{02}_t(\mathbf{i})$ is nonincreasing in i_1 from Theorem 3.2(a), and $z^{12}_t(\mathbf{i} - e_{01})$ is nondecreasing in i_1 from Theorem 3.2(a3), it must follow that $z^{01}_t(\mathbf{i})$ is nonincreasing in i_1. Similarly, because $z^{02}_t(\mathbf{i})$ is nonincreasing in i_2 from Theorem 3.2(a1), and $z^{01}_t(\mathbf{i})$ is nondecreasing in i_2 from Theorem 3.2(a2), it must follow that $z^{12}_t(\mathbf{i} - e_{01})$ is nonincreasing in i_2. Therefore, the assertions (a) hold.

(b) Immediately from (a) and Lemma 2.1(d). □

Proof of Theorem 3.4. (a) We shall use techniques similar to those in Lee and Hersh (1993). First, consider the case of $m = 1$. We can rewrite $v_t(\mathbf{i}) - v_{t-1}(\mathbf{i})$ as

$$v_t(\mathbf{i}) - v_{t-1}(\mathbf{i}) = (v_t(\mathbf{i} - e_{j,j+1}) + \gamma_t) - (v_{t-1}(\mathbf{i} - e_{j,j+1}) + \gamma_{t-1}),$$

where γ_t is the expected revenue that can be generated by selling one additional seat in flight leg j from the present time t through the deadline. Note that this seat can be sold to a request for one seat of any trip including the flight leg j or as part of a multiple seat. Because the expected revenue from selling one additional seat within t decision periods is at least as much as the expected revenue from selling the seat within $t - 1$ decision periods, it is obvious that $\gamma_t \geqslant \gamma_{t-1}$. Hence, it implies that

$$v_t(\mathbf{i}) - v_{t-1}(\mathbf{i}) \geqslant v_t(\mathbf{i} - e_{j,j+1}) - v_{t-1}(\mathbf{i} - e_{j,j+1}).$$

Thus, we have

$$z^{j,j+1}_{t1}(\mathbf{i}) \geqslant z^{j,j+1}_{t-1,1}(\mathbf{i}). \tag{26}$$

Note that, for any given $m \geqslant 2$, we get

$$\begin{aligned} v_t(\mathbf{i}) - v_t(\mathbf{i} - me_{j,j+1}) &= (v_t(\mathbf{i}) - v_t(\mathbf{i} - e_{j,j+1})) \\ &\quad + (v_t(\mathbf{i} - e_{j,j+1}) - v_t(\mathbf{i} - 2e_{j,j+1})) + \cdots \\ &\quad + (v_t(\mathbf{i} - (m-1)e_{j,j+1}) - v_t(\mathbf{i} - me_{j,j+1})) \\ &\geqslant (v_{t-1}(\mathbf{i}) - v_{t-1}(\mathbf{i} - e_{j,j+1})) \\ &\quad + (v_{t-1}(\mathbf{i} - e_{j,j+1}) - v_{t-1}(\mathbf{i} - 2e_{j,j+1})) + \cdots \\ &\quad + (v_{t-1}(\mathbf{i} - (m-1)e_{j,j+1}) - v_{t-1}(\mathbf{i} - me_{j,j+1})) \\ &= v_{t-1}(\mathbf{i}) - v_{t-1}(\mathbf{i} - me_{j,j+1}). \end{aligned}$$

Thus, we have

$$z^{j,j+1}_{tm}(\mathbf{i}) \geqslant z^{j,j+1}_{t-1,m}(\mathbf{i}) \tag{27}$$

for $m \geqslant 2$. Consequently, it follows from inequalities 26 and 27 that $z^{j,j+1}_{tm}(\mathbf{i})$ is nondecreasing in t for any j, m and $\mathbf{i}$. Furthermore, from Eq. 3 we get

$$\begin{aligned} z^{jk}_{tm}(\mathbf{i}) &= (v_t(\mathbf{i}) - v_{t-1}(\mathbf{i} - me_{jk}))/m \\ &= (v_t(\mathbf{i}) - v_t(\mathbf{i} - me_{j,j+1}))/m \\ &\quad + (v_t(\mathbf{i} - me_{j,j+1}) - v_{t-1}(\mathbf{i} - me_{j,j+2}))/m \\ &\quad + \cdots + (v_t(\mathbf{i} - me_{j,k-1}) - v_t(\mathbf{i} - me_{jk}))/m \\ &= z^{j,j+1}_{tm}(\mathbf{i}) + z^{j+1,j+2}_{tm}(\mathbf{i} - me_{j,j+1}) + \cdots \\ &\quad + z^{k-1,k}_{tm}(\mathbf{i} - me_{j,k-1}) \\ &\geqslant z^{j,j+1}_{t-1,m}(\mathbf{i}) + z^{j+1,j+2}_{t-1,m}(\mathbf{i} - me_{j,j+1}) + \cdots \\ &\quad + z^{k-1,k}_{t-1,m}(\mathbf{i} - me_{j,k-1}) \\ &= (v_{t-1}(\mathbf{i}) - v_{t-1}(\mathbf{i} - me_{j,j+1}))/m \\ &\quad + (v_{t-1}(\mathbf{i} - me_{j,j+1}) - v_{t-1}(\mathbf{i} - me_{j,j+2}))/m \\ &\quad + \cdots + (v_{t-1}(\mathbf{i} - me_{j,k-1}) - v_{t-1}(\mathbf{i} - me_{jk}))/m \\ &= z^{jk}_{t-1,m}(\mathbf{i}). \end{aligned}$$

Consequently, it follows that $z^{jk}_{tm}(\mathbf{i})$ is nondecreasing in t.

(b) Immediately follows from (a) and Lemma 2.1(d). □

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