DECISIONS UNDER UNCERTAINTY, CERTAINTY EQUIVALENTS, AND GENERALIZED ENTROPY

Jointly with Aharon Ben-Tal and Marc Teboulle

Attitude towards wealth	Utility
Attitude towards risk	Expected utility (EU)
Axioms for EU	Risk aversion (RA)
Generalizations of EU	Dual theory
Bad predictions of EU	Arrow-Pratt RA indices
Certainty equivalents	The Recourse CE (RCE)

$$S_v(\mathbf{Z}) \ := \ \sup_z \{z \ + \ \mathsf{E}\, v(\mathbf{Z} - z)\}$$

Meaning of $v(\cdot)$	Recoverability
Economic models	Production, investment, insurance
Extremal principles	RCE, EU and ϕ–divergence

$$I_\phi(\mathbf{p}, \mathbf{q}) \ := \ \sum q_i \, \phi\left(\frac{p_i}{q_i}\right)$$

The **St. Petersburg paradox**
Daniel Bernoulli (1738)

RV $\mathbf{Z}$: Prob $\left\{\mathbf{Z} = 2^{n-1}\right\} = \frac{1}{2^n}$, $(n = 1, 2, \dots)$
The **expected value** of $\mathbf{Z}$:

$$\mathsf{E}\,\mathbf{Z} = \sum_{n=1}^{\infty} 2^{n-1}\,\frac{1}{2^n} = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots = \infty$$

Why "paradox"?

Game with two players, A and B (= Bank).
A fair coin $\{H, T\}$ is tossed until H appears.
If on trial n, B pays A : 2^{n-1}\$, $(n = 1, 2, \dots)$
The **value** of the game for A is the RV $\mathbf{Z}$.
What is a fair **ticket price** ? $\mathsf{E}\,\mathbf{Z}$?
Hence the "paradox".

Bernoulli introduced **moral expectation**:
The **expected value** to A is:

$$\sum_{n=1}^{\infty} u\left(2^{n-1}\right)\frac{1}{2^n}$$

where $u(W)$ is A's **utility** of **wealth** W.

Bernoulli's **logarithmic utility**

If **wealth** increases from W to $W + \Delta W$, the **utility** increases by Δu

$$\Delta u \;=\; K \,\frac{\Delta W}{W}$$

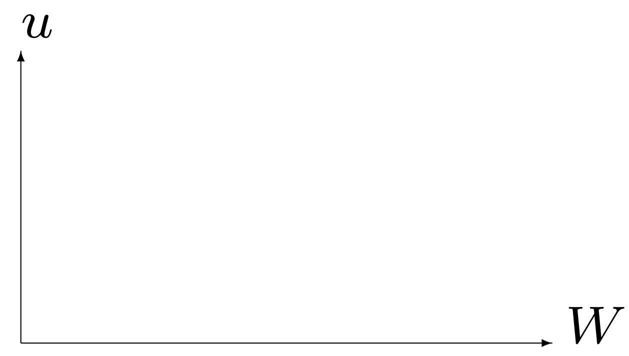
$$u(W) \;=\; \int_c^W \; K\; \frac{d\,W}{W} \;=\; K \,\log\, \frac{W}{c}$$

Back to St Petersburg: A's EU is:

$$\begin{aligned} \mathsf{E}\, u(\mathbf{Z}) \;&=\; K \sum_{n=1}^{\infty} \log\left(2^{n-1}\right) \frac{1}{2^n} \\ &=\; K \sum_{n=1}^{\infty} \frac{n-1}{2^n} \;=\; K \end{aligned}$$

Attitude towards wealth

Marginal utility $\searrow$ $\iff$ u **concave**



The **Weber–Fechner law** (1860)

The **response** to a **stimulus** diminishes with each repetition of that stimulus within some specified time period.

Sensation is a logarithmic function of stimulus

Axioms for EU theory
von Neumann-Morgenstern (1947)

Preferences between RV's $\mathbf{X}$, $\mathbf{Y}$:

$\mathbf{X} \sim \mathbf{Y}$	**indifference**
$\mathbf{X} \succeq \mathbf{Y}$	$\mathbf{X}$ **better** than $\mathbf{Y}$
$\mathbf{X} \preceq \mathbf{Y}$	$\mathbf{X}$ **worse** than $\mathbf{Y}$

Completeness: For any $\mathbf{X}$, $\mathbf{Y}$,

either $\mathbf{X} \succeq \mathbf{Y}$ or $\mathbf{X} \preceq \mathbf{Y}$.

Transitivity: $\mathbf{X} \succeq \mathbf{Y}$ & $\mathbf{Y} \succeq \mathbf{Z} \implies \mathbf{X} \succeq \mathbf{Z}$.

Continuity: Given $\mathbf{X} \succeq \mathbf{Y} \succeq \mathbf{Z}$,
there exists a probability p in $[0,1]$ such that

$$\mathbf{Y} \sim p\,\mathbf{X} + (1-p)\,\mathbf{Z}$$

Independence: If $\mathbf{X} \succeq \mathbf{Y}$, then
for any $p \in [0,1]$ and any $\mathbf{Z}$,

$$p\,\mathbf{X} + (1-p)\,\mathbf{Z} \succeq p\mathbf{Y} + (1-p)\,\mathbf{Z}$$

Existence of utility

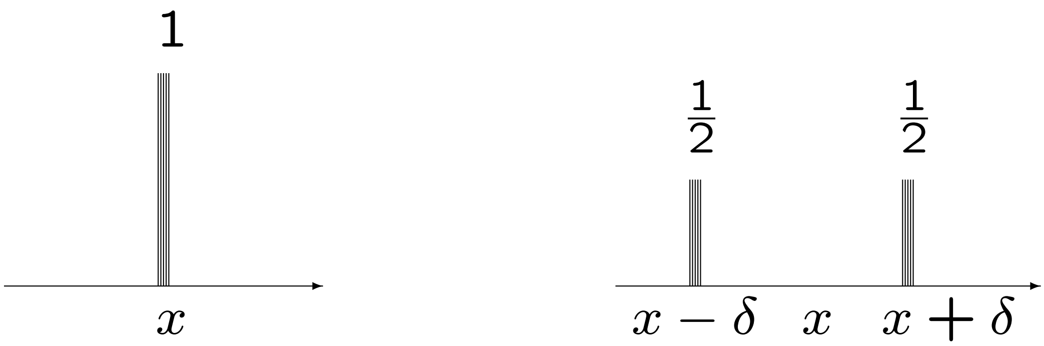
The axioms $\Longrightarrow$ existence of a **utility** u, (continuous, nondecreasing) such that

$$\mathbf{X} \succeq \mathbf{Y} \iff \mathsf{E}\, u(\mathbf{X}) \geq \mathsf{E}\, u(\mathbf{Y})$$

The utility u is unique up to a positive linear transformation: For any $\alpha > 0$, β, the utilities

$$u \text{ and } \alpha u + \beta$$

represent the same $\succeq$



$$u(x) \quad \geq \quad \frac{1}{2}\, u(x-\delta) + \frac{1}{2}\, u(x+\delta)$$

Concavity of u $\iff$ **Risk Aversion**

The **Allais paradox** (1953)
Kahneman and Tversky (1979)

Choose between:

A: 4,000 (prob. .80) 0 (prob. .20)	**B**: 3,000 (certainty)
[20%]	[80%]

C: 4,000 (prob. .20) 0 (prob. .80)	**D**: 3,000 (prob. .25) 0 (prob. .75)
[65%]	[35%]

$$\mathbf{B} \succeq \mathbf{A} \Longrightarrow u(3,000) > .80\,u(4,000)$$
$$\mathbf{C} \succeq \mathbf{D} \Longrightarrow .25\,u(3,000) < .20\,u(4,000)$$

Stochastic programming with recourse
2–stage stoch. prog., Dantzig (1955)

$$\max \{f(z) : \ g(z) \ \leq \ \mathbf{Z}\}$$

$z =$ **decision variable** $\qquad f(\cdot) =$ **profit**

$\mathbf{Z} =$ **budget** $\qquad g(\cdot) =$ **consumption**

If $\mathbf{Z}$ is RV, the a priori decision z may violate

$$g(z) \ \leq \ \mathbf{Z}$$

Let $\quad y =$ **second stage decision**, with

$h(y) =$ **consumption**, $\qquad v(y) =$ **profit**

$$\max_z \ \{f(z) \ + \ \mathsf{E}\left(\max_y \ \{v(y) : g(z) + h(y) \ \leq \ \mathbf{Z}\}\right)\}$$

If z, y are scalars, $h(\cdot) \nearrow$, $v(\cdot) \nearrow$,

$$\max_z \ \{f(z) \ + \ \mathsf{E}\left(\max_y\{v(y) : g(z) + y \leq \mathbf{Z}\}\right)\}$$

$$\max_z \ \{f(z) \ + \ \mathsf{E}\, v(\mathbf{Z} - g(z))\}$$

Recourse certainty equivalent (RCE)

The **value** of a RV $\mathbf{Z}$

$$\text{value of } \mathbf{Z} = \max\{z : z \leq \mathbf{Z}\}$$

Stoch. Prog. with Recourse interpretation:

$$\sup_z \{z + \mathsf{E}\, v(\mathbf{Z} - z)\}$$

$v(x)$ = current (before realization) value of x

$v(\cdot)$ = the **value-risk function**

Another interpretation:

$\mathbf{Z}$ = future income (RV)

z = **loan** against $\mathbf{Z}$ (unrestricted)

$v(\cdot)$ = **utility**

Assumptions on $v(\cdot)$

(v1) $v(0) = 0$
(v2) $v(\cdot)$ is strictly increasing
(v3) $v(x) \le x$ for all x
(v4) $v(\cdot)$ is strictly concave
(v5) v is continuously differentiable

$$\text{(v1) and (v2)} \implies v(x) < 0 \quad \text{for } x < 0,$$

$\therefore$ $v(\cdot)$ = **penalty function** for

$$z \le \mathbf{Z}$$

Special class of value-risk functions

$$\mathcal{U} := \left\{ v : \begin{array}{c} v \text{ strict.} \nearrow, \quad \text{strict. concave} \\ \text{cont. diff.}, \quad v(0) = 0, \quad v'(0) = 1 \end{array} \right\}$$

the **normalized utility functions**.

For concave v the gradient inequality

$$v(x) \le v(0) + v'(0)x$$

shows that all $v \in \mathcal{U}$ satisfy (v3).

Attainment of $\sup_z \{z + \mathsf{E}\, v(\mathbf{Z} - z)\}$

$$\mathcal{U} \;=\; \left\{ v : \begin{array}{c} v \text{ strict. } \nearrow, \quad \text{strict. concave} \\ \text{cont. diff.,} \quad v(0) = 0, \quad v'(0) = 1 \end{array} \right\}$$

Lemma. Let the RV $\mathbf{Z}$ have finite support

$$\left[z_{\text{min}}, \quad z_{\text{max}} \right] .$$

Then $\quad \forall\ v \in \mathcal{U} \quad$ the supremum in RCE is attained uniquely for some z^*,

$$z_{\text{min}} \;\leq\; z^* \;\leq\; z_{\text{max}}$$

which is the solution of

$$\mathsf{E}\, v'(\mathbf{Z} - z^*) \;=\; 1$$

so that

$$S_v(\mathbf{Z}) \;=\; z^* \;+\; \mathsf{E}\, v(\mathbf{Z} - z^*)$$

Proof. $\mathbf{Z} - z_{\text{min}} \;\geq\; 0$ with probability 1.
Also $v'(\cdot) \;\searrow$,

$$\therefore \quad \mathsf{E}\, v'\left(\mathbf{Z} - z_{\text{min}}\right) \;\leq\; \mathsf{E}\, v'(0) \;=\; 1$$

$$\text{Similarly} \quad \mathsf{E}\, v'\left(\mathbf{Z} - z_{\text{max}}\right) \;\geq\; \mathsf{E}\, v'(0) \;=\; 1$$

Properties of $S_v(\mathbf{Z}) \;:=\; \sup_z \{z \;+\; \mathsf{E}\, v(\mathbf{Z} - z)\}$

Shift additivity. For any $v : \mathbf{R} \to \mathbf{R}$, any RV $\mathbf{Z}$ and any constant c

$$S_v(\mathbf{Z} + c) \;=\; S_v(\mathbf{Z}) \;+\; c$$

For any function $v : \mathbf{R} \to \mathbf{R}$,

$$\begin{aligned} S_v(\mathbf{Z} + c) \;&=\; \sup_z \{z \;+\; \mathsf{E}\, v(\mathbf{Z} + c - z)\} \\ &=\; c \;+\; \sup_z \{(z - c) \;+\; \mathsf{E}\, v(\mathbf{Z} - (z - c))\} \\ &=\; c \;+\; S_v(\mathbf{Z}) \end{aligned}$$

For any RV $\mathbf{Z}$, constant S, if

$$\mathbf{Z} \;\sim\; S$$

then

$$\mathbf{Z} + c \;\sim\; S + c$$

for all constant c.

(v1) $v(0) = 0$
(v2) $v(\cdot)$ is strictly increasing
(v3) $v(x) \leq x$ for all x
(v4) $v(\cdot)$ is strictly concave
(v5) v is continuously differentiable

Consistency. If v satisfies (v1), (v3) then,

$$S_v(c) = c, \quad \forall \text{ constant } c$$

Subhomogeneity. If v satisfies (v1) and (v4) then, for any RV $\mathbf{Z}$

$$\frac{1}{\lambda} S_v(\lambda \mathbf{Z}) \text{ is decreasing in } \lambda \text{ , } \lambda > 0$$

$$\begin{aligned} S_v(\lambda \mathbf{Z}) &\leq \lambda S_v(\mathbf{Z}) \text{ , } \forall \lambda > 1 \\ \lambda \mathbf{Z} &\preceq \lambda S_v(\mathbf{Z}) \\ \text{since } \mathsf{E}\,(\lambda \mathbf{Z}) &= \lambda \mathbf{Z} \\ \text{Var } (\lambda \mathbf{Z}) &= \lambda^2 \text{Var } \mathbf{Z} > \lambda \text{Var } \mathbf{Z} \quad \text{if } \lambda > 1 \end{aligned}$$

An interesting result: For $v \in \mathcal{U}$,

$$\lim_{\lambda \to 0+} \frac{1}{\lambda} S_v(\lambda \mathbf{Z}) = \mathsf{E}\,\mathbf{Z}$$

(v1) $v(0) = 0$
(v2) $v(\cdot)$ is strictly increasing
(v3) $v(x) \leq x$ for all x
(v4) $v(\cdot)$ is strictly concave
(v5) v is continuously differentiable

Monotonicity. If v satisfies (v2) then, for any RV $\mathbf{X}$ and any nonnegative RV $\mathbf{Y}$,

$$S_v(\mathbf{X} + \mathbf{Y}) \geq S_v(\mathbf{X})$$

If v satisfies (v1) and (v2), and if

$$\mathbf{Z} \geq z_{\text{min}} \text{ with probability 1,}$$

then

$$S_v(\mathbf{Z}) \geq z_{\text{min}}$$

Take:
$$\mathbf{X} = z_{\text{min}} \text{ (degenerate RV)}$$
$$\mathbf{Y} = \mathbf{Z} - z_{\text{min}}$$

(v1) $v(0) = 0$
(v2) $v(\cdot)$ is strictly increasing
(v3) $v(x) \leq x$ for all x
(v4) $v(\cdot)$ is strictly concave
(v5) v is continuously differentiable

$$\mathcal{U} \;:=\; \left\{ v : \begin{array}{c} v \text{ strict. } \nearrow, \quad \text{strict. concave} \\ \text{cont. diff.}, \quad v(0) = 0, \quad v'(0) = 1 \end{array} \right\}$$

Risk aversion. v satisfies (v3) iff

$$S_v(\mathbf{Z}) \;\leq\; \mathsf{E}\,\mathbf{Z} \quad \text{for all RV's } \mathbf{Z}$$

Concavity. For any $v \in \mathcal{U}$, RV's $\mathbf{X}_0$, $\mathbf{X}_1$ and $0 < \alpha < 1$,

$$S_v(\alpha\mathbf{X}_1 + (1-\alpha)\mathbf{X}_0) \;\geq\; \alpha S_v(\mathbf{X}_1) + (1-\alpha) S_v(\mathbf{X}_0)$$

2^{nd} order stochastic dominance. Let $\mathbf{X}$, $\mathbf{Y}$ be RV's with compact support. Then

$$S_v(\mathbf{X}) \;\geq\; S_v(\mathbf{Y}) \quad \text{for all } v \in \mathcal{U}$$

if and only if

$$\mathsf{E}\,v(\mathbf{X}) \;\geq\; \mathsf{E}\,v(\mathbf{Y}) \quad \text{for all } v \in \mathcal{U}$$

Exponential value-risk function

$$u(z) \ := \ 1 - e^{-z}$$

The optimality condition $\mathsf{E}\, u'(\mathbf{Z} - z^*) \ = \ 1$ gives $\mathsf{E}\, e^{-\mathbf{Z}+z^*} = 1$ or $z^* \ = \ -\log \ \mathsf{E}\, e^{-\mathbf{Z}}$

$$\therefore \quad S_u(\mathbf{Z}) \ = \ -\log \ \mathsf{E}\, e^{-\mathbf{Z}}$$

For the **exponential utility** $u(z) = 1 - e^{-z}$,

$$u^{-1}\mathsf{E}\, u(\mathbf{Z}) \ = \ -\log \ \mathsf{E}\, e^{-\mathbf{Z}}$$

Quadratic value-risk function

$$u(z) \ := \ z - \frac{1}{2} z^2 , \quad z \le 1$$

For RV $\mathbf{Z}$: $z_{\max} \le 1$, $\mathsf{E}\,\mathbf{Z} \ = \ \mu$, $\mathsf{Var}\,\mathbf{Z} \ = \ \sigma^2$,

$$z^* \ = \ \mu, \ S_u(\mathbf{Z}) \ = \ \mu - \frac{1}{2}\sigma^2$$

In both exponential & quadratic $u(\cdot)$

$$S_u\left(\sum_{i=1}^{n} \mathbf{Z}_i\right) \ = \ \sum_{i=1}^{n} \ S_u(\mathbf{Z}_i)$$

for independent RV's $\{\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_n\}$

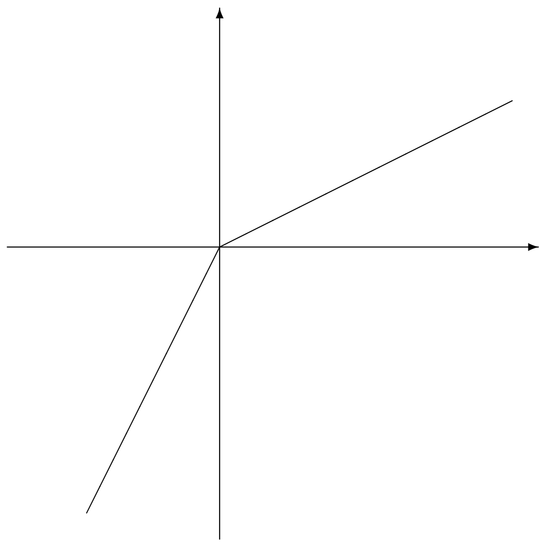
The hybrid model

u = exponential utility, $\mathbf{Z} \sim \mathrm{N}(\mu, \sigma^2)$,

$$S_u(\mathbf{Z}) = \mu - \frac{1}{2}\sigma^2$$

Piecewise linear value-risk function

$$v(t) = \begin{cases} \beta t\,, & t \leq 0 \\ \alpha t\,, & t > 0 \end{cases} \quad , \; 0 < \alpha < 1 < \beta$$



$$z^* = F^{-1}\left(\frac{1-\alpha}{\beta-\alpha}\right)$$

where F is the c.d.f. of $\mathbf{Z}$.

$$S_v(\mathbf{Z}) = \beta \int^{z^*} t\,dF(t) + \alpha \int_{z^*} t\,dF(t).$$

Convexity

$$\mathbf{P}^n \;:=\; \left\{ \mathbf{p} \in \mathbf{R}^n_+ \;:\; \sum_{i=1}^{n} p_i = 1 \right\}$$

For any $\mathbf{x} = (x_i) \in \mathbf{R}^n$, $\mathbf{p} = (p_i) \in \mathbf{P}^n$, let $[\mathbf{x}, \mathbf{p}]$ denote the RV $\mathbf{X}$, Prob $\{\mathbf{X} = x_i\} = p_i$

The RCE of $[\mathbf{x}, \mathbf{p}]$ is

$$S_v([\mathbf{x}, \mathbf{p}]) \;=\; \max_z \left\{ z \;+\; \sum_{i=1}^{n} p_i \, v(x_i - z) \right\}$$

For any $v : \mathbf{R} \to \mathbf{R}$, and $\mathbf{x} \in \mathbf{R}^n$,

$$S_v([\mathbf{x}, \mathbf{p}]) \text{ is convex in } \mathbf{p}$$

For v concave, and any $\mathbf{p} \in \mathbf{P}^n$,

$$S_v([\mathbf{x}, \mathbf{p}]) \text{ is concave in } \mathbf{x}$$

	Function of $\mathbf{p}$	Function of $\mathbf{x}$
Eu, u concave	linear	concave
$u^{-1}Eu$, u concave	convex	?
Y_f	convex	linear
S_v	convex	concave (if v)

Recoverability & meaning of $v(\cdot)$

Let $\mathbf{X} = (x, p)$ denote the RV

$$\mathbf{X} = \begin{cases} x, & \text{with probability } p \\ 0, & \text{with probability } \bar{p} = 1 - p \end{cases}$$

$$S_v(x,p) = \sup_z \{z + pv(x-z) + \bar{p}v(-z)\}$$

Theorem. If $v \in \mathcal{U}$ then

$$v(x) = \frac{\partial}{\partial p} S_v(x,p)\Big|_{p=0}$$

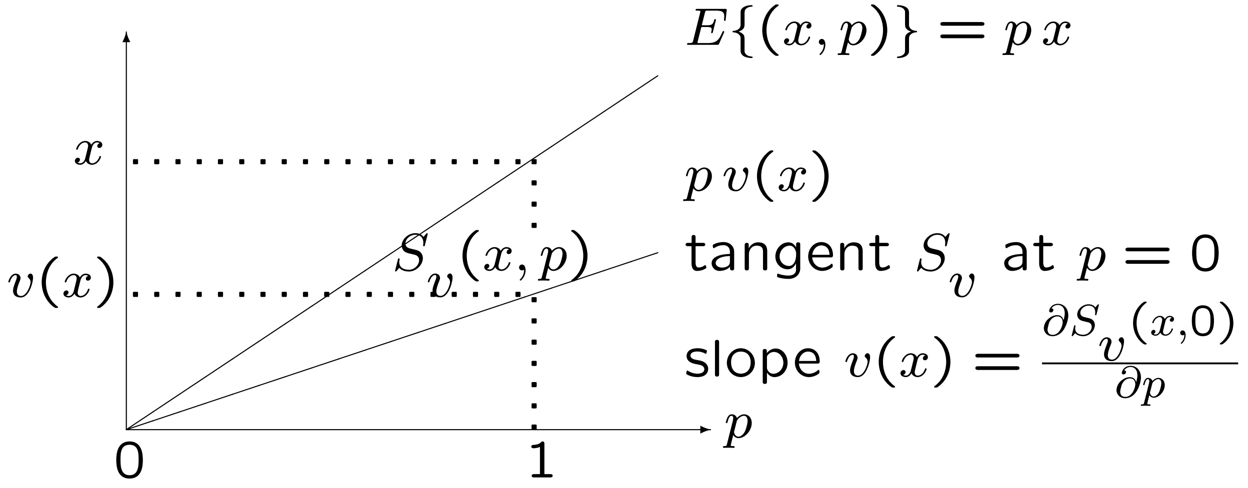
$$p = \frac{\partial}{\partial x} S_v(x,p)\Big|_{x=0}$$

Going from $(x, 0)$ to (x, p) changes the RCE by

$$\Delta(x,p) = S_v(x,p) - S_v(x,0)$$

The **rate of change** is

$$\frac{S_v(x,p) - S_v(x,0)}{p} \approx v(x) \quad \text{for} \quad p \ll 1$$



Risk functions

For any certainty equivalent CE(x,p),

$\frac{\partial}{\partial p}CE(x,0)$ **value risk function** of CE

$\frac{\partial}{\partial x}CE(0,p)$ **probability risk function** of CE

CE(x,p)	$\frac{\partial}{\partial p}CE(x,0)$	$\frac{\partial}{\partial x}CE(0,p)$
$C_u = u^{-1}Eu$	$\frac{u(x)-u(0)}{u'(0)}$	p
M_u	$\frac{u(x)-u(0)}{u'(0)}$	p
S_v	$v(x)$	p

where $M_u(\mathbf{Z})$ is defined by

$$\mathsf{E}\,u\Big(\mathbf{Z} - M_u(\mathbf{Z})\Big) \;=\; 0$$

Strong risk aversion
Rothschild-Stiglitz (1970,1971)
Diamond-Stiglitz (1974)

In EU Theory, **risk aversion** of EU, i.e.

$$\mathsf{E}\, u(\mathbf{Z}) \;\le\; \mathsf{E}\,\mathbf{Z}\,, \quad \forall \text{ RV } \mathbf{Z}$$

is equivalent to the concavity of $u(\cdot)$.

$$S_v(\mathbf{Z}) \;:=\; \sup_z \{z \;+\; \mathsf{E}\, v(\mathbf{Z} - z)\} \qquad \text{(RCE)}$$

In RCE Theory, **risk aversion** of S_v, i.e.

$$S_v(\mathbf{Z}) \;\le\; \mathsf{E}\,\mathbf{Z}\,, \quad \forall \text{ RV } \mathbf{Z}$$

is equivalent to

$$v(x) \;\le x, \quad \forall\; x$$

What corresponds to $v(\cdot)$ concave ?

Risk increases

Let $F_{\mathbf{X}}$, $F_{\mathbf{Y}}$ be the c.d.f.'s of RV $\mathbf{X}$, $\mathbf{Y}$ with support $[a, b]$.

If $\mathsf{E}\,\mathbf{X} = \mathsf{E}\,\mathbf{Y}$ and there is a $c \in [a, b]$ such that

$$\begin{aligned} F_{\mathbf{Y}}(t) &\leq F_{\mathbf{X}}(t), \quad a \leq t \leq c \\ F_{\mathbf{X}}(t) &\geq F_{\mathbf{Y}}(t), \quad c \leq t \leq b \end{aligned}$$

then $F_{\mathbf{Y}}$ **differs** from $F_{\mathbf{X}}$ by a **mean preserving simple increase in risk** (MPSIR).

$F_{\mathbf{Y}}$ **differs** from $F_{\mathbf{X}}$ by a **mean preserving increase in risk** (MPIR) if it differs from $F_{\mathbf{X}}$ by a sequence of MPSIR's.

A CE maximizing DM exhibits **strong risk aversion** if

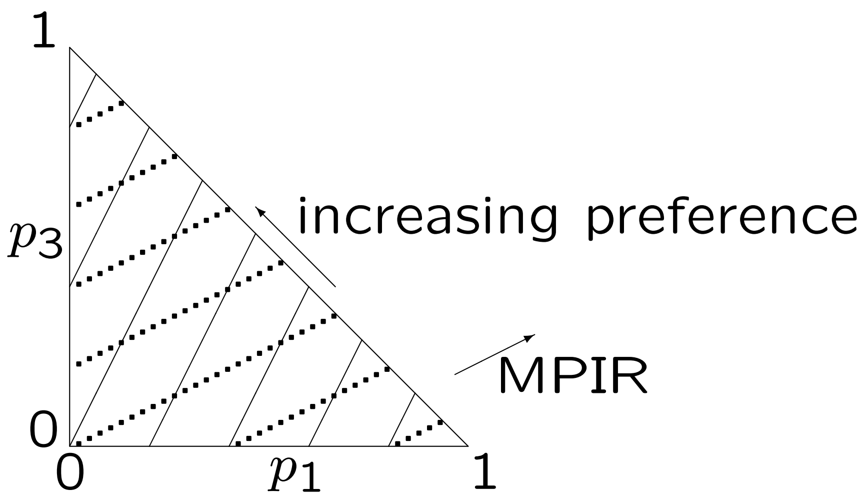
$$\left\{ \begin{array}{c} F_{\mathbf{Y}} \text{ differs from } F_{\mathbf{X}} \\ \text{by a MPIR} \end{array} \right\} \Longrightarrow \text{CE}(\mathbf{Y}) \leq \text{CE}(\mathbf{X})$$

Strong risk aversion in EU Theory

Fix $x_1 < x_2 < x_3$, and let

$$D\{x_1, x_2, x_3\} := \begin{cases} \textbf{probability distributions} \\ \text{over} \quad x_1, x_2, x_3 \end{cases}$$

Each $\mathbf{p} = (p_1, p_2, p_3) \in D\{x_1, x_2, x_3\}$ is represented as a point in a unit triangle in the (p_1, p_3)-plane, where $p_2 = 1 - p_1 - p_3$



Iso-mean and iso-EU lines in $D\{x_1, x_2, x_3\}$

$$\textbf{Strong RA} \iff \frac{\overbrace{u(x_2) - u(x_1)}^{\text{slope iso-EU}}}{u(x_3) - u(x_2)} > \frac{\overbrace{x_2 - x_1}^{\text{slope iso-mean}}}{x_3 - x_2}$$

Functionals & approximations

RV $\vec{\mathbf{Z}} = (\mathbf{Z}_i) \in \mathbf{R}^n$, $\quad \mathsf{E}\,\vec{\mathbf{Z}} = \vec{\boldsymbol{\mu}}$, $\quad \mathsf{Cov}\,\vec{\mathbf{Z}} = \Sigma$.
For any $\vec{\mathbf{y}} \in \mathbf{R}^n$

$$s_u(\vec{\mathbf{y}}) \;:=\; S_u\left(\vec{\mathbf{y}} \cdot \vec{\mathbf{Z}}\right)$$

If $u \in \mathcal{U} \cap \mathcal{C}^{(2)}$, then
(a) The functional $s_u(\cdot)$ is concave, and given by

$$s_u(\vec{\mathbf{y}}) \;=\; z^*(\vec{\mathbf{y}}) \;+\; Eu(\vec{\mathbf{y}} \cdot \vec{\mathbf{Z}} - z^*(\vec{\mathbf{y}}))$$

where $z^*(\vec{\mathbf{y}})$ is the unique solution z of

$$E\,u'(\vec{\mathbf{y}} \cdot \vec{\mathbf{Z}} - z) \;=\; 1$$

(b) Moreover,

$$s_u(\vec{\mathbf{0}}) = 0\,, \;\; \nabla s_u(\vec{\mathbf{0}}) = \vec{\boldsymbol{\mu}}, \;\;\; \nabla^2 s_u(\vec{\mathbf{0}}) = u''(0)\Sigma$$
$$z^*(\vec{\mathbf{0}}) = 0\;, \;\; \nabla z^*(\vec{\mathbf{0}}) = \vec{\boldsymbol{\mu}}$$

and if $u \in \mathcal{C}^{(3)}$,

$$\nabla^2 z^*(\vec{\mathbf{0}}) = \frac{u'''(0)}{u''(0)}\Sigma$$

$$s_u(\vec{\mathbf{y}}) \;=\; \vec{\boldsymbol{\mu}} \cdot \vec{\mathbf{y}} \;+\; \frac{1}{2}u''(0)\vec{\mathbf{y}} \cdot \Sigma\vec{\mathbf{y}} \;+\; \circ\left(\|\,\vec{\mathbf{y}}\,\|^2\right)$$

$$s_u(\mathbf{y}) \;=\; \mu\cdot\mathbf{y} \;+\; \tfrac{1}{2}u''(0)\mathbf{y}\cdot\Sigma\mathbf{y} \;+\; \circ\left(\|\,\mathbf{y}\,\|^2\right)$$

For $n = 1$ and $y = 1$,

$$\begin{aligned} S_u(\mathbf{Z}) &\approx \mu + \frac{1}{2}u''(0)\,\sigma^2 \\ &= \mu - \frac{1}{2}r(0)\,\sigma^2 \\ \text{where} \quad r(z) &= -\frac{u''(z)}{u'(z)} \end{aligned}$$

the **Arrow-Pratt absolute risk-aversion index**

The approximation is exact if

(i) u is quadratic, or
(ii) u is exponential, $\mathbf{Z}$ is normal

Comparing with the Taylor expansion of the classical CE

$$\begin{aligned} c_u(y) &:= u^{-1}Eu(y\cdot\mathbf{Z}) \\ \text{we get} \quad c_u(y) - s_u(y) &= \circ(\|\,y\,\|^2) \end{aligned}$$

Competitive firm under price uncertainty
Sandmo (1971)

$$\Pi(q) \;=\; qP - c(q) - B$$

$P =$ **price** (RV) $q =$ **production**
$B =$ **fixed cost** $c(q) =$ **variable cost**
$\Pi =$ **profit**

Assume:

$$c(0) = 0, \quad c' > 0, \quad c'' > 0$$

$$u \in \mathcal{U}$$

$$\max_{q \geq 0} \quad \mathsf{E}\, u(\Pi(q)) \qquad \text{(EU Model)}$$

$$\max_{q \geq 0} \quad S_u(\Pi(q)) \qquad \text{(RCE Model)}$$

$$\Pi(q) \;=\; qP - c(q) - B$$

EU	RCE
$q^* > 0$ iff $\mu > c'(0)$	same
$q^* < q_{\text{cer}}$	same
$q^* \nearrow$ as $B \nearrow$ if $r \nearrow$ $q^* \searrow$ as $B \nearrow$ if $r \searrow$	q^* is independent of B, $\forall u$
$q^* \nearrow$ as $P \to P + \epsilon$ if $r \searrow$	$q^* \nearrow$ as $P \to P + \epsilon$ $\forall u \in \mathcal{U}$
Profit tax $(1-t)\Pi(q)$	Profit tax $(1-t)\Pi(q)$
$q^* \nearrow$ with t if $r =$ const and $R \nearrow$ or $r \searrow$ and $R \nearrow$ $r \searrow$ and $R =$ const	$q^* \nearrow$ with t for all $u \in \mathcal{U}$

$$r(z) \;:=\; -\frac{u''(z)}{u'(z)}\,, \qquad R(z) \;:=\; zr(z)$$

Investment in risky & safe assets
Arrow (1971)

$$Y(a) = A - a + (1 + X)a = A + aX$$

$A =$ **initial wealth**, $Y =$ **final wealth**.
$X =$ **rate of return** of risky investment
$a =$ **amount invested in risky assets**

EU	RCE
$a^* > 0$ iff $\mathrm{E}\, X > 0$	same
Wealth elasticity of cash balance $m = A - a^*$ $\frac{\mathrm{d}\, m/\mathrm{d}\, A}{m/A} \geq 1$ if $R \nearrow$	$\frac{\mathrm{d}\, m/\mathrm{d}\, A}{m/A} \geq 1 \ \forall u \in \mathcal{U}$
$X \to X + h \implies a^*(h) \nearrow$	same
$X \to (1 + h)X$ $\implies a^*(h) = \frac{a^*}{1 + h}$ Tobin (1958)	same

Investment in risky & safe assets
Cass-Stiglitz (1970), Fishburn-Porter (1976)

$$Y(k) \;=\; k\,W_0\,X \;+\; (1-k)\,W_0\,\rho$$

$W_0 =$ **initial wealth**
$X =$ **rate of return, risky asset**
$\rho =$ **rate of return, safe asset**
$k =$ **fraction invested in risky asset**

EU	RCE
If $0 < k^* < 1$ $\rho \nearrow \implies (1-k^*) \nearrow$ if $r \nearrow$ or $R \leq 1$	Conclusion holds for all $u \in \mathcal{U}$

Insurance
Ehrlich-Becker (1972)
Lippman-McCall (1981)

n	**states of nature**	
$\mathbf{p}$	$=$	$(p_1, \ldots, p_n)$ their **probabilities**
$\bar{q}_i$	$=$	**premium** for 1$ coverage in state i
$\bar{B}$	$=$	**insurance budget**
q_i	$=$	$\bar{q}_i / \sum_{j=1}^n \bar{q}_j$ = **normalized premium**
B	$=$	$\bar{B} / \sum_{j=1}^n \bar{q}_j$ = **normalized budget**
x_i	$=$	**income** in state i
$\mathbf{x}$	$=$	$(x_1, \ldots, x_n)$ **decision variable**

$$\sum_{i=1}^n q_i\, x_i \;=\; B \qquad \textbf{budget constraint}$$

The **optimal value** is

$$\begin{aligned} I^* &= \max_{\mathbf{x}} \left\{ S_v([\mathbf{x}, \mathbf{p}]) \;:\; \sum_{i=1}^n q_i\, x_i \;=\; B \right\} \\ &= \max_{\mathbf{x},\ \sum q_i x_i \,=\, B} \;\; \max_z \left\{ z \;+\; \sum_{i=1}^n p_i\, v(x_i - z) \right\} \\ &= S_v([\mathbf{x}^*, \mathbf{p}]) \end{aligned}$$

$\mathbf{x}^* \;=\; (x_i^*)$ is the **optimal insurance coverage**.

Insurace: The solution

$$
\begin{aligned}
I^* &= \max_{\mathbf{x}} \{S_v([\mathbf{x}, \mathbf{p}]) \ : \ \sum_{i=1}^{n} q_i \, x_i \ = \ B\} \\
&= \max_{\mathbf{x}, \ \sum q_i x_i \ = \ B} \ \max_{z} \{z \ + \ \sum_{i=1}^{n} p_i \, v(x_i - z)\} \\
&= S_v([\mathbf{x}^*, \mathbf{p}])
\end{aligned}
$$

The optimal insurance coverage is

$$x_i^* \ = \ B \ + \ \phi\left(\frac{q_i}{p_i}\right) - \sum_{j=1}^{n} q_j \, \phi\left(\frac{q_j}{p_j}\right)$$

where

$$\phi \ = \ \left(v'\right)^{-1}$$

Moreover,

$$I^* \ = \ B - \sum q_i \, \phi\left(\frac{q_i}{p_i}\right) \ + \ \sum p_i \, v(\phi\left(\frac{q_i}{p_i}\right))$$

The ϕ–divergence functional
Csiszár (1978)

Given a convex function $\phi : \mathbf{R}_+ \to \mathbf{R}$,
the ϕ–**divergence functional**,

$$I_\phi(\mathbf{p}, \mathbf{q}) \ := \ \sum_{i=1}^{n} q_i \, \phi\left(\frac{p_i}{q_i}\right)$$

is a "distance function" on $\mathbf{P}^n$.

$\phi : \mathbf{R}_+ \to \mathbf{R}$ convex
$\implies \ I_\phi(\mathbf{p}, \mathbf{q})$ jointly convex in $\mathbf{p}$ and $\mathbf{q}$.

Let $\phi : \mathbf{R}_+ \to \mathbf{R}$ be convex.
Then for any $\mathbf{p}, \mathbf{q} \in \mathbf{R}^n_{++}$,

$$I_\phi(\mathbf{p}, \mathbf{q}) \ \geq \ \left(\sum_{i=1}^{n} q_i\right) \phi\left(\frac{\sum_{i=1}^{n} p_i}{\sum_{i=1}^{n} q_i}\right)$$

If ϕ is strictly convex, equality holds iff

$$\frac{p_1}{q_1} = \frac{p_2}{q_2} = \cdots = \frac{p_n}{q_n}$$

Adjoints

Let $\phi : \mathbf{R}_+ \to \mathbf{R}$ be convex, $\phi(1) = 0$.
The **adjoint** of ϕ is

$$\phi^\diamond(t) := t\,\phi\left(\frac{1}{t}\right), \quad t > 0.$$

(a) $\phi^{\diamond\,\diamond} = \phi$.
(b) $\phi^\diamond(\cdot)$ is convex.
(c) $\phi^\diamond(1) = 0$
(d) For any $\mathbf{p}, \mathbf{q} \in \mathbf{R}^n_{++}$

$$I_\phi(\mathbf{p}, \mathbf{q}) = I_{\phi^\diamond}(\mathbf{q}, \mathbf{p})$$

(e) $\phi^\diamond(t)$ is increasing for $t > 1$.

In general, $I_\phi(\mathbf{p}, \mathbf{q}) \neq I_\phi(\mathbf{q}, \mathbf{p})$.
Also $I_\phi(\cdot, \cdot)$ does not satisfy triangle inequality.
To get symmetry use:

$$I_\phi(\mathbf{p}, \mathbf{q}) + I_{\phi^\diamond}(\mathbf{p}, \mathbf{q})$$

The Kullback-Leibler distance

$$
\begin{aligned}
I_\phi(\mathbf{p},\mathbf{q}) &= \sum_{i=1}^{n} p_i \log \frac{p_i}{q_i} \\
\text{from} \quad \phi(t) &:= t \log t, \quad t > 0, \\
\phi^\diamond &= t \left(\frac{1}{t} \log \frac{1}{t}\right) = -\log t
\end{aligned}
$$

In particular, for $\mathbf{p} = \left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right)$

$$
\begin{aligned}
I_{\phi^\diamond}(\mathbf{p},\mathbf{q}) &= -\sum_{i=1}^{n} q_i \log \frac{1}{nq_i} \\
&= \sum_{i=1}^{n} q_i \log q_i + \log n
\end{aligned}
$$

The Hellinger distance

$$
\begin{aligned}
I_\phi(\mathbf{p},\mathbf{q}) &= \sum_{i=1}^{n} \left(\sqrt{p_i} - \sqrt{q_i}\right)^2 \\
\text{from} \quad \phi(t) &:= \left(1 - \sqrt{t}\right)^2, \quad t > 0 \\
\phi^\diamond &= \phi
\end{aligned}
$$

α–order entropy

For $\alpha > 1$ let,

$$\phi(t) := t^{\alpha}, \quad t > 0$$

$$\therefore \quad I_{\phi}(\mathbf{p},\mathbf{q}) = \sum_{i=1}^{n} (p_i)^{\alpha} (q_i)^{1-\alpha}$$

the α-**order entropy**.

$$\phi^{\diamond}(t) = t^{1-\alpha}$$

The χ^2–distance

$$\phi(t) := (t-1)^2, \quad t > 0,$$

$$I_{\phi}(\mathbf{p},\mathbf{q}) = \sum_{i=1}^{n} \left(\frac{p_i^2 - q_i^2}{q_i} \right)$$

the χ^2-**distance**.

$$\phi^{\diamond}(t) = (\sqrt{t} - \frac{1}{\sqrt{t}})^2$$

$$= t + \frac{1}{t} - 2$$

The variation distance

$$\phi(t) \;:=\; |\, t-1 \,|, \quad t>0,$$
$$I_\phi(\mathbf{p},\mathbf{q}) \;=\; \sum_{i=1}^{n} |\, p_i - q_i \,|$$

Indicator function

Let $\alpha < 1 < \beta$ and define

$$\phi(t) \;:=\; \delta(t \mid [\alpha,\beta]) \;=\; \begin{cases} 0 & \text{if } \alpha \le t \le \beta \,, \\ \infty & \text{otherwise} \,, \end{cases}$$

indicator function of the interval $[\alpha,\beta]$,

$$\phi^{\diamond}(t) \;=\; \delta\left(t \mid \left[\frac{1}{\beta},\frac{1}{\alpha}\right]\right)$$
$$I_\phi(\mathbf{p},\mathbf{q}) \;=\; \begin{cases} 0 & \text{if } \alpha \le \frac{p_i}{q_i} \le \beta \,, \\ & \qquad \forall i = 1,\ldots,n \\ \infty & \text{otherwise} \,. \end{cases}$$

Convex analysis

For $f : \mathbf{R}^n \to \mathbf{R}$,

$$f^*(y) := \sup_x \{\mathbf{y} \cdot \mathbf{x} - f(\mathbf{x})\} \quad \textbf{convex conjugate}$$
$$f_*(y) := \inf_x \{\mathbf{y} \cdot \mathbf{x} - f(\mathbf{x})\} \quad \textbf{concave conjugate}$$

(a) ϕ convex $\implies (-\phi)_*(x) = -\phi^*(-x)\,,\ \forall x\,,$

(b) u concave $\implies (-u)^*(x) = -u_*(-x)\,,\ \forall x\,.$

If $u : \mathbf{R} \to \mathbf{R}$ strictly $\nearrow$ & concave then

(a) dom $u_* \subset \mathbf{R}_+$

(b) dom $(u^{-1})^* \subset \mathbf{R}_+$

(c) $y \in$ dom $(u^{-1})^* \iff \frac{1}{y} \in$ dom u_*

(d) For all $y > 0$

$$(u^{-1})^*(y) = -y\, u_* \left(\frac{1}{y}\right) = -(u_*)^\diamond(y)$$

Let $\phi : \mathbf{R} \to \mathbf{R}$ be convex. Then

(a) dom $\phi = \mathbf{R}_+ \implies \phi^*$ is nondecreasing

(b) dom $\phi = \mathbf{R}_{++} \implies \phi^*$ is increasing

A special functional

For $h : \mathbf{R}^n \to \mathbf{R}$ define $h_+ : \mathbf{R}^n \to \mathbf{R}$ by

$$h_+(\mathbf{y}) := \sup_{\substack{z \in \mathbf{R} \\ \mathbf{x} \in \mathbf{R}^n \\ \mathbf{y} \cdot \mathbf{x} = 0}} \{z + h(\mathbf{x} - z\mathbf{e})\}$$

Then for any $\mathbf{y} \in \mathbf{R}^n$ with $\sum y_i \neq 0$,

$$h_+(\mathbf{y}) = -h_* \left(\frac{\mathbf{y}}{\sum y_i} \right)$$

If h is strictly concave and $\mathcal{C}^{(1)}$,

$$\begin{aligned} z^* &= -\mathbf{v}^T \mathbf{u}^* = -\frac{\mathbf{y}^T}{\mathbf{e} \cdot \mathbf{y}} (\nabla h)^{-1} \left(\frac{\mathbf{y}}{\mathbf{e} \cdot \mathbf{y}} \right) \\ \mathbf{x}^* &= \mathbf{u}^* + z^* \mathbf{e} = \left(I - \frac{\mathbf{e}\mathbf{y}^T}{\mathbf{e}^T \mathbf{y}} \right) (\nabla h)^{-1} \left(\frac{\mathbf{y}}{\mathbf{e} \cdot \mathbf{y}} \right) \\ &= P_{\mathbf{e}^\perp} (\nabla h)^{-1} \left(\frac{\mathbf{y}}{\mathbf{e} \cdot \mathbf{y}} \right) \end{aligned}$$

$P_{\mathbf{e}^\perp}$:= orthogonal projection $\perp \mathbf{e}$.

$$-z^* \mathbf{e} = \frac{\mathbf{e}\mathbf{y}^T}{\mathbf{e}^T \mathbf{y}} (\nabla h)^{-1} \left(\frac{\mathbf{y}}{\mathbf{e} \cdot \mathbf{y}} \right) = P_{\mathbf{e}} (\nabla h)^{-1} \left(\frac{\mathbf{y}}{\mathbf{e} \cdot \mathbf{y}} \right)$$

Extremal principles for RCE, EU

If $u : \mathbf{R} \to \mathbf{R}$ is strictly $\nearrow$ & concave, then

$$\phi \; := \; -u_*$$

is convex and $\phi : \mathbf{R}_+ \to \mathbf{R}$. Also

$$\phi^\diamond \; = \; (u^{-1})^*$$

Conversely, for convex $\phi : \mathbf{R}_+ \to \mathbf{R}$,

$$u(t) \; := \; -\phi^*(-t)$$

is concave.

For any RV $\mathbf{X} = [\mathbf{x}, \mathbf{p}]$,

$$\begin{aligned} S_u([\mathbf{x},\mathbf{p}]) &= \inf_{\mathbf{q} \in \mathbf{P}^n} \left\{ I_\phi(\mathbf{q},\mathbf{p}) \; + \; \sum_{i=1}^n q_i \, x_i \right\} \\ &= \inf_{\mathbf{q} \in \mathbf{P}^n} \left\{ I_{\phi^\diamond}(\mathbf{p},\mathbf{q}) \; + \; \sum_{i=1}^n q_i \, x_i \right\} \\ \mathsf{E}\, u([\mathbf{x},\mathbf{p}]) &= \inf_{\mathbf{q} \in \mathbf{R}_+^n} \left\{ I_\phi(\mathbf{q},\mathbf{p}) \; + \; \sum_{i=1}^n q_i \, x_i \right\} \end{aligned}$$

Duality

$$S_u([\mathbf{x},\mathbf{p}]) \;=\; \inf_{\mathbf{q}\in\mathbf{P}^n} \left\{ I_\phi(\mathbf{q},\mathbf{p}) \;+\; \sum_{i=1}^{n} q_i\, x_i \right\}$$

The RCE, in LHS, is the optimal value of

(P) $\sup\, \{z : \text{“}z \;\le\; \mathbf{X}\text{”}\}$

in the sense of **recourse optimization**.

The RHS is the “dual” of (P),

(D) $\inf\, \{z : \text{“}z \;\ge\; \mathbf{X}\text{”}\}$

using a **stochastic penalty function** $P(\cdot)$ to “enforce” the stochastic constraint

$$\text{“}z \;\ge\; \mathbf{X}\text{”}$$

(D1) $\inf\, \{z \;+\; P(z)\}$

Stochastic penalty

Given the RV $\mathbf{X} = [\mathbf{x}, \mathbf{p}]$ and z, define

$$R\{z\} := \left\{ \mathbf{q} \in \mathbf{P}^n : \; z \geq \sum_i q_i \, x_i \right\},$$

representing the RV's $\{\mathbf{Z} := [\mathbf{x}, \mathbf{q}] : \; \mathbf{q} \in R\{z\}\}$ supported as $\mathbf{X}$, satisfying $z \geq \mathbf{X}$ "in the mean". $P(z)$ is the **penalty**

$$P(z) := \text{"dist"} \, (\mathbf{p}, R\{z\}) = \inf_{\mathbf{q} \in R\{z\}} \text{"dist"} \, (\mathbf{q}, \mathbf{p})$$

and "dist" is induced by the ϕ–divergence,

$$P(z) = \inf_{\mathbf{q} \in R\{z\}} I_\phi(\mathbf{q}, \mathbf{p})$$

$$\inf_{\substack{z \\ \mathbf{q} \in R\{z\}}} \{z + I_\phi(\mathbf{q}, \mathbf{p})\} =$$

$$= \inf_{\mathbf{q} \in \mathbf{P}^n} \inf_{z \geq \sum q_i x_i} \{z + I_\phi(\mathbf{q}, \mathbf{p})\}$$

$$= \inf_{\mathbf{q} \in \mathbf{P}^n} \{I_\phi(\mathbf{q}, \mathbf{p}) + \sum q_i \, x_i\}$$

Extremal principles for ϕ–divergence

Let $\phi : \mathbf{R}_+ \to \mathbf{R}$ convex, dom $\phi \subset \mathbf{R}_{++}$. Define

$$u(t) := -\phi^*(-t)$$

Then for any $\mathbf{p}, \mathbf{q} \in \mathbf{P}^n$,

$$I_\phi(\mathbf{q}, \mathbf{p}) = \sup_{\substack{\mathbf{x} \in \mathbf{R}^n \\ \mathbf{q} \cdot \mathbf{x} = 0}} S_u([\mathbf{x}, \mathbf{p}])$$

Furthermore, if dom $\phi = \mathbf{R}_{++}$, define the function

$$v(x) := (\phi^*)^{-1}(x),$$

Then

$$I_\phi(\mathbf{p}, \mathbf{q}) = \sup_{\substack{\mathbf{x} \in \mathbf{R}^n \\ \mathbf{q} \cdot \mathbf{x} = 0}} S_v([\mathbf{x}, \mathbf{p}])$$

For all $\mathbf{p} \in \mathbf{P}^n$, $\mathbf{q} \in \mathbf{R}^n_+$,

$$I_\phi(\mathbf{q}, \mathbf{p}) = \sup_{\mathbf{x} \in \mathbf{R}^n} \left\{ \sum_{i=1}^n p_i \, u(x_i) - \sum_{i=1}^n q_i \, x_i \right\}$$