# DECISIONS UNDER UNCERTAINTY, CERTAINTY EQUIVALENTS, AND GENERALIZED ENTROPY

Jointly with Aharon Ben-Tal and Marc Teboulle

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\[ S_v(Z) := \sup_z \{ z + E v(Z - z) \} \]

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\[ I_{\phi}(p, q) := \sum q_i \phi \left( \frac{p_i}{q_i} \right) \]
The **St. Petersburg paradox**
Daniel Bernoulli (1738)

RV $Z$ : \( \text{Prob}\ \{Z = 2^{n-1}\} = \frac{1}{2^n}, \ (n = 1, 2, \ldots) \)

The **expected value** of $Z$:

\[
E Z = \sum_{n=1}^{\infty} 2^{n-1} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots = \infty
\]

Why “paradox”?

---

**Game** with two players, $A$ and $B$ (= Bank).
A fair coin \( \{H, T\} \) is tossed until $H$ appears.
If on trial $n$, $B$ pays $A$ : $2^{n-1} \$ , \((n = 1, 2, \ldots)\)
The **value** of the game for $A$ is the RV $Z$.
What is a fair **ticket price** ? $E Z$ ?
Hence the “paradox”.

---

Bernoulli introduced **moral expectation**:
The **expected value** to $A$ is:

\[
\sum_{n=1}^{\infty} \left(2^{n-1}\right) u \left(\frac{1}{2^n}\right)
\]

where $u(W)$ is $A$’s **utility** of wealth $W$. 

Bernoulli’s **logarithmic utility**

If **wealth** increases from $W$ to $W + \Delta W$, the **utility** increases by $\Delta u$

$$\Delta u = K \frac{\Delta W}{W}$$

$$u(W) = \int_c^W K \frac{dW}{W} = K \log \frac{W}{c}$$

Back to St Petersburg: $A$'s EU is:

$$E_u(Z) = K \sum_{n=1}^{\infty} \log(2^{n-1}) \frac{1}{2^n}$$
$$= K \sum_{n=1}^{\infty} \frac{n-1}{2^n} = K$$
Attitude towards wealth

Marginal utility $\downarrow \iff u$ concave

\[ u \]
\[ W \]

The Weber–Fechner law (1860)

The response to a stimulus diminishes with each repetition of that stimulus within some specified time period.

Sensation is a logarithmic function of stimulus
Preferences between RV’s $X$, $Y$:

- $X \sim Y$ \textit{indifference}
- $X \succeq Y$ \textit{$X$ better than $Y$}
- $X \preceq Y$ \textit{$X$ worse than $Y$}

\underline{Completeness}: For any $X$, $Y$,

either $X \succeq Y$ or $X \preceq Y$.

\underline{Transitivity}: $X \succeq Y$ \& $Y \succeq Z \implies X \succeq Z$.

\underline{Continuity}: Given $X \succeq Y \succeq Z$,
there exists a probability $p$ in $[0, 1]$ such that

$$Y \sim pX + (1-p)Z$$

\underline{Independence}: If $X \succeq Y$, then
for any $p \in [0, 1]$ and any $Z$,

$$pX + (1-p)Z \succeq pY + (1-p)Z$$
Existence of utility

The axioms $\iff$ existence of a utility $u$, (continuous, nondecreasing) such that

$$X \succeq Y \iff \mathbb{E} u(X) \geq \mathbb{E} u(Y)$$

The utility $u$ is unique up to a positive linear transformation: For any $\alpha > 0, \beta$, the utilities $u$ and $\alpha u + \beta$ represent the same $\succeq$

\[ u(x) \geq \frac{1}{2} u(x - \delta) + \frac{1}{2} u(x + \delta) \]

**Concavity of $u \iff$ Risk Aversion**
Choose between:

**A**: 4,000 (prob. .80)  
0 (prob. .20)  

[20%]

**B**: 3,000 (certainty)  

[80%]

**C**: 4,000 (prob. .20)  
0 (prob. .80)  

[65%]

**D**: 3,000 (prob. .25)  
0 (prob. .75)  

[35%]

\[ \text{B} \preceq \text{A} \quad \text{implies} \quad u(3,000) > .80 \, u(4,000) \]

\[ \text{C} \preceq \text{D} \quad \text{implies} \quad .25 \, u(3,000) < .20 \, u(4,000) \]
Stochastic programming with recourse
2–stage stoch. prog., Dantzig (1955)

\[ \max \{ f(z) : g(z) \leq Z \} \]

\( z = \text{decision variable} \quad f(\cdot) = \text{profit} \)

\( Z = \text{budget} \quad g(\cdot) = \text{consumption} \)

If \( Z \) is RV, the a priori decision \( z \) may violate

\[ g(z) \leq Z \]

Let \( y = \text{second stage decision} \), with

\( h(y) = \text{consumption} \quad v(y) = \text{profit} \)

\[ \max_z \{ f(z) + \mathbb{E} \left( \max_y \{ v(y) : g(z) + h(y) \leq Z \} \right) \} \]

If \( z, y \) are scalars, \( h(\cdot) \nearrow, \ v(\cdot) \nearrow, \)

\[ \max_z \{ f(z) + \mathbb{E} \left( \max_y \{ v(y) : g(z) + y \leq Z \} \right) \} \]

\[ \max_z \{ f(z) + \mathbb{E} v(Z - g(z)) \} \]
Recourse certainty equivalent (RCE)

The value of a RV $Z$

$$\text{value of } Z = \max \{ z : z \leq Z \}$$

Stoch. Prog. with Recourse interpretation:

$$\sup_z \{ z + E_v(Z - z) \}$$

$v(x) = \text{current (before realization) value of } x$

$v(\cdot) = \text{the value-risk function}$

Another interpretation:

$Z = \text{future income (RV)}$

$z = \text{loan against } Z \text{ (unrestricted)}$

$v(\cdot) = \text{utility}$
Assumptions on $v(\cdot)$

(v1) $v(0) = 0$
(v2) $v(\cdot)$ is strictly increasing
(v3) $v(x) \leq x$ for all $x$
(v4) $v(\cdot)$ is strictly concave
(v5) $v$ is continuously differentiable

(v1) and (v2) $\implies v(x) < 0$ for $x < 0$,

$\therefore v(\cdot) = \text{penalty function}$ for $z \leq Z$

Special class of value-risk functions

$$\mathcal{U} := \left\{ v : \quad v \text{ strict. } \nearrow, \quad \text{strict. concave cont. diff.,} \quad v(0) = 0, \quad v'(0) = 1 \right\}$$

the normalized utility functions.

For concave $v$ the gradient inequality

$$v(x) \leq v(0) + v'(0)x$$

shows that all $v \in \mathcal{U}$ satisfy (v3).
**Lemma.** Let the RV $Z$ have finite support $[z_{\text{min}}, z_{\text{max}}]$. Then $\forall v \in \mathcal{U}$ the supremum in RCE is attained uniquely for some $z^*$, 

$$z_{\text{min}} \leq z^* \leq z_{\text{max}}$$

which is the solution of

$$E v'(Z - z^*) = 1$$

so that

$$S_v(Z) = z^* + E v(Z - z^*)$$

**Proof.** $Z - z_{\text{min}} \geq 0$ with probability 1. Also $v'(\cdot) \searrow$,

$$\therefore E v'(Z - z_{\text{min}}) \leq E v'(0) = 1$$

Similarly $E v'(Z - z_{\text{max}}) \geq E v'(0) = 1$
Properties of $S_v(Z) := \sup_z \{ z + E v(Z - z) \}$

---

**Shift additivity.** For any $v : \mathbb{R} \rightarrow \mathbb{R}$, any RV $Z$ and any constant $c$

$$S_v(Z + c) = S_v(Z) + c$$

For any function $v : \mathbb{R} \rightarrow \mathbb{R}$,

$$S_v(Z + c) = \sup_z \{ z + E v(Z + c - z) \}$$

$$= c + \sup_z \{ (z - c) + E v(Z - (z - c)) \}$$

$$= c + S_v(Z)$$

For any RV $Z$, constant $S$, if

$$Z \sim S$$

then

$$Z + c \sim S + c$$

for all constant $c$. 

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(v1) \( v(0) = 0 \)
(v2) \( v(\cdot) \) is strictly increasing
(v3) \( v(x) \leq x \) for all \( x \)
(v4) \( v(\cdot) \) is strictly concave
(v5) \( v \) is continuously differentiable

**Consistency.** If \( v \) satisfies (v1), (v3) then,

\[
S_v(c) = c, \quad \forall \text{ constant } c
\]

**Subhomogeneity.** If \( v \) satisfies (v1) and (v4) then, for any RV \( Z \)

\[
\frac{1}{\lambda} S_v(\lambda Z) \text{ is decreasing in } \lambda, \quad \lambda > 0
\]

\[
S_v(\lambda Z) \leq \lambda S_v(Z), \quad \forall \lambda > 1
\]

\[
\lambda Z \leq \lambda S_v(Z)
\]

since \( E(\lambda Z) = \lambda Z \)

\[
\text{Var}(\lambda Z) = \lambda^2 \text{Var}(Z) > \lambda \text{Var}(Z) \quad \text{if } \lambda > 1
\]

An interesting result: For \( v \in \mathcal{U} \),

\[
\lim_{\lambda \to 0^+} \frac{1}{\lambda} S_v(\lambda Z) = E(Z)
\]
(v1) $v(0) = 0$
(v2) $v(\cdot)$ is strictly increasing
(v3) $v(x) \leq x$ for all $x$
(v4) $v(\cdot)$ is strictly concave
(v5) $v$ is continuously differentiable

---

**Monotonicity.** If $v$ satisfies (v2) then, for any RV $X$ and any nonnegative RV $Y$,

$$S_v(X + Y) \geq S_v(X)$$

---

If $v$ satisfies (v1) and (v2), and if

$$Z \geq z_{\min} \text{ with probability 1},$$

then

$$S_v(Z) \geq z_{\min}$$

Take:

$$X = z_{\min} \text{ (degenerate RV)}$$
$$Y = Z - z_{\min}$$
(v1) $v(0) = 0$
(v2) $v(\cdot)$ is strictly increasing
(v3) $v(x) \leq x$ for all $x$
(v4) $v(\cdot)$ is strictly concave
(v5) $v$ is continuously differentiable

$$
U := \left\{ v : \begin{array}{c}
v \text{ strict. } \nearrow, \text{ strict. concave} \\
\text{cont. diff., } v(0) = 0, \quad v'(0) = 1
\end{array} \right\}
$$

**Risk aversion.** $v$ satisfies (v3) iff

$$
S_v(Z) \leq E Z \quad \text{for all RV's } Z
$$

**Concavity.** For any $v \in U$, RV's $X_0$, $X_1$ and $0 < \alpha < 1$,

$$
S_v(\alpha X_1 + (1 - \alpha) X_0) \geq \alpha S_v(X_1) + (1 - \alpha) S_v(X_0)
$$

**2nd order stochastic dominance.** Let $X$, $Y$ be RV's with compact support. Then

$$
S_v(X) \geq S_v(Y) \quad \text{for all } v \in U
$$

if and only if

$$
E v(X) \geq E v(Y) \quad \text{for all } v \in U
$$
Exponential value-risk function

\[ u(z) := 1 - e^{-z} \]

The optimality condition \( E u'(Z - z^*) = 1 \) gives \( E e^{-Z+z^*} = 1 \) or \( z^* = -\log E e^{-Z} \)

\[ \therefore S_u(Z) = -\log E e^{-Z} \]

For the exponential utility \( u(z) = 1 - e^{-z} \),

\[ u^{-1}E u(Z) = -\log E e^{-Z} \]

Quadratic value-risk function

\[ u(z) := z - \frac{1}{2} z^2, \quad z \leq 1 \]

For RV \( Z \) : \( z_{\text{max}} \leq 1, \ E Z = \mu, \ \text{Var} Z = \sigma^2, \)

\[ z^* = \mu, \ S_u(Z) = \mu - \frac{1}{2} \sigma^2 \]

In both exponential & quadratic \( u(\cdot) \)

\[ S_u \left( \sum_{i=1}^{n} Z_i \right) = \sum_{i=1}^{n} S_u(Z_i) \]

for independent RV's \( \{Z_1, Z_2, \ldots, Z_n\} \)
The hybrid model

\[ u = \text{exponential utility}, \ Z \sim N(\mu, \sigma^2), \]

\[ S_u(Z) = \mu - \frac{1}{2}\sigma^2. \]

Piecewise linear value-risk function

\[ v(t) = \begin{cases} \beta t, & t \leq 0 \\ \alpha t, & t > 0 \end{cases}, \ 0 < \alpha < 1 < \beta. \]

\[ z^* = F^{-1} \left( \frac{1 - \alpha}{\beta - \alpha} \right) \]

where \( F \) is the c.d.f. of \( Z. \)

\[ S_v(Z) = \beta \int_{z^*}^{\infty} tdF(t) + \alpha \int_{z^*}^{\infty} tdF(t). \]
Convexity

\[ P^n := \left\{ p \in \mathbb{R}^n_+ : \sum_{i=1}^n p_i = 1 \right\} \]

For any \( x = (x_i) \in \mathbb{R}^n, \ p = (p_i) \in P^n, \) let \([x, p]\) denote the RV \( X, \) \( \text{Prob} \{X = x_i\} = p_i \)

The RCE of \([x, p]\) is

\[ S_v([x, p]) = \max_z \left\{ z + \sum_{i=1}^n p_i v(x_i - z) \right\} \]

For any \( v: \mathbb{R} \to \mathbb{R}, \) and \( x \in \mathbb{R}^n, \)

\( S_v([x, p]) \) is convex in \( p \)

For \( v \) concave, and any \( p \in P^n, \)

\( S_v([x, p]) \) is concave in \( x \)

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<th>Function of ( p )</th>
<th>Function of ( x )</th>
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<td>Eu, ( u ) concave</td>
<td>concave</td>
</tr>
<tr>
<td>( u^{-1}Eu, \ u ) concave</td>
<td>convex linear</td>
</tr>
<tr>
<td>( Y_f )</td>
<td>convex</td>
</tr>
<tr>
<td>( S_v )</td>
<td>convex (if ( v ))</td>
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Let $X = (x, p)$ denote the RV

$$X = \begin{cases} x, \text{ with probability } p \\ 0, \text{ with probability } \bar{p} = 1 - p \end{cases}$$

$$S_v(x, p) = \sup_z \{ z + pv(x - z) + \bar{p}v(-z) \}$$

**Theorem.** If $v \in \mathcal{U}$ then

$$v(x) = \left. \frac{\partial}{\partial p} S_v(x, p) \right|_{p=0}$$

$$p = \left. \frac{\partial}{\partial x} S_v(x, p) \right|_{x=0}$$

Going from $(x, 0)$ to $(x, p)$ changes the RCE by

$$\Delta(x, p) = S_v(x, p) - S_v(x, 0)$$

The **rate of change** is

$$\frac{S_v(x, p) - S_v(x, 0)}{p} \approx v(x) \text{ for } p \ll 1$$
\[ E\{(x, p)\} = px \]

\[ pv(x) \]

tangent \( S_v \) at \( p = 0 \)

\[ \text{slope } v(x) = \frac{\partial S_v(x,0)}{\partial p} \]

---

**Risk functions**

For any certainty equivalent \( CE(x, p) \),

\[ \frac{\partial}{\partial p} CE(x, 0) \quad \text{value risk function of } CE \]

\[ \frac{\partial}{\partial x} CE(0, p) \quad \text{probability risk function of } CE \]

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<tr>
<th>( CE(x, p) )</th>
<th>( \frac{\partial}{\partial p} CE(x, 0) )</th>
<th>( \frac{\partial}{\partial x} CE(0, p) )</th>
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<tr>
<td>( C_u = u^{-1} Eu )</td>
<td>( \frac{u(x) - u(0)}{u'(0)} )</td>
<td>( p )</td>
</tr>
<tr>
<td>( M_u )</td>
<td>( \frac{u(x) - u(0)}{u'(0)} )</td>
<td>( p )</td>
</tr>
<tr>
<td>( S_v )</td>
<td>( v(x) )</td>
<td>( p )</td>
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where \( M_u(Z) \) is defined by

\[ E u \left( Z - M_u(Z) \right) = 0 \]
Strong risk aversion
Diamond-Stiglitz (1974)

In EU Theory, **risk aversion** of EU, i.e.

\[ E u(Z) \leq E Z, \quad \forall \text{ RV } Z \]

is equivalent to the concavity of \( u(\cdot) \).

\[
S_v(Z) := \sup_z \{ z + E v(Z - z) \} \quad \text{(RCE)}
\]

In RCE Theory, **risk aversion** of \( S_v \), i.e.

\[ S_v(Z) \leq E Z, \quad \forall \text{ RV } Z \]

is equivalent to

\[ v(x) \leq x, \quad \forall x \]

What corresponds to \( v(\cdot) \) concave?
Risk increases

Let $F_X$, $F_Y$ be the c.d.f.'s of RV $X$, $Y$ with support $[a, b]$.

If $E_X = E_Y$ and there is a $c \in [a, b]$ such that

\[
F_Y(t) \leq F_X(t), \quad a \leq t \leq c \\
F_X(t) \geq F_Y(t), \quad c \leq t \leq b
\]

then $F_Y$ differs from $F_X$ by a mean preserving simple increase in risk (MPSIR).

$F_Y$ differs from $F_X$ by a mean preserving increase in risk (MPIR) if it differs from $F_X$ by a sequence of MPSIR’s.

A CE maximizing DM exhibits strong risk aversion if

\[
\left\{ \begin{array}{l}
F_Y \text{ differs from } F_X \\
\text{by a MPIR}
\end{array} \right\} \implies CE(Y) \leq CE(X)
\]
Strong risk aversion in EU Theory

Fix $x_1 < x_2 < x_3$, and let

$$D\{x_1, x_2, x_3\} := \left\{ \text{probability distributions} \atop \text{over} \ x_1, x_2, x_3 \right\}$$

Each $p = (p_1, p_2, p_3) \in D\{x_1, x_2, x_3\}$ is represented as a point in a unit triangle in the $(p_1, p_3)$-plane, where $p_2 = 1 - p_1 - p_3$

Iso-mean and iso-EU lines in $D\{x_1, x_2, x_3\}$

**Strong RA** $\iff$ 

$$\frac{u(x_2) - u(x_1)}{u(x_3) - u(x_2)} > \frac{x_2 - x_1}{x_3 - x_2}$$
Functionals & approximations

RV $\tilde{Z} = (Z_i) \in \mathbb{R}^n$, \quad $E \tilde{Z} = \bar{\mu}$, \quad $\text{Cov} \tilde{Z} = \Sigma$.
For any $\vec{y} \in \mathbb{R}^n$

$$s_u(\vec{y}) := S_u (\vec{y} \cdot \tilde{Z})$$

If $u \in \mathcal{U} \cap \mathcal{C}^2$, then

(a) The functional $s_u(\cdot)$ is concave, and given by

$$s_u(\vec{y}) = z^*(\vec{y}) + E u (\vec{y} \cdot \tilde{Z} - z^*(\vec{y}))$$

where $z^*(\vec{y})$ is the unique solution $z$ of

$$E u'(\vec{y} \cdot \tilde{Z} - z) = 1$$

(b) Moreover,

$$s_u(\vec{0}) = 0, \quad \nabla s_u(\vec{0}) = \bar{\mu}, \quad \nabla^2 s_u(\vec{0}) = u''(0) \Sigma$$

$$z^*(\vec{0}) = 0, \quad \nabla z^*(\vec{0}) = \bar{\mu}$$

and if $u \in \mathcal{C}^3$,

$$\nabla^2 z^*(\vec{0}) = \frac{u'''(0)}{u''(0)} \Sigma$$

$$s_u(\vec{y}) = \bar{\mu} \cdot \vec{y} + \frac{1}{2} u''(0) \vec{y} \cdot \Sigma \vec{y} + o(\| \vec{y} \|^2)$$

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\[ s_u(y) = \mu \cdot y + \frac{1}{2} u''(0) y \cdot \Sigma y + o(\|y\|^2) \]

For \( n = 1 \) and \( y = 1 \),

\[
S_u(Z) \approx \mu + \frac{1}{2} u''(0) \sigma^2 \\
= \mu - \frac{1}{2} r(0) \sigma^2
\]

where \( r(z) = -\frac{u''(z)}{u'(z)} \)

the **Arrow-Pratt absolute risk-aversion index**

The approximation is exact if

(i) \( u \) is quadratic, or
(ii) \( u \) is exponential, \( Z \) is normal

Comparing with the Taylor expansion of the classical CE

\[ c_u(y) := u^{-1} Eu(y \cdot Z) \]

we get \[ c_u(y) - s_u(y) = o(\|y\|^2) \]
Competitive firm under price uncertainty
Sandmo (1971)

\[ \Pi(q) = qP - c(q) - B \]

\[ P = \text{price (RV)} \quad q = \text{production} \]
\[ B = \text{fixed cost} \quad c(q) = \text{variable cost} \]
\[ \Pi = \text{profit} \]

Assume:
\[ c(0) = 0, \quad c' > 0, \quad c'' > 0 \]
\[ u \in U \]

\[ \max_{q \geq 0} \quad \mathbb{E} u(\Pi(q)) \quad \text{(EU Model)} \]
\[ \max_{q \geq 0} \quad S'_u(\Pi(q)) \quad \text{(RCE Model)} \]
\[ \Pi(q) = qP - c(q) - B \]

<table>
<thead>
<tr>
<th>EU</th>
<th>RCE</th>
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</thead>
<tbody>
<tr>
<td>( q^* &gt; 0 ) iff ( \mu &gt; c'(0) )</td>
<td>same</td>
</tr>
<tr>
<td>( q^* &lt; q_{\text{cer}} )</td>
<td>same</td>
</tr>
<tr>
<td>( q^* \uparrow ) as ( B \uparrow ) if ( r \uparrow )</td>
<td>( q^* ) is independent of ( B ), ( \forall u )</td>
</tr>
<tr>
<td>( q^* \downarrow ) as ( B \uparrow ) if ( r \downarrow )</td>
<td>( q^* \uparrow ) as ( P \rightarrow P + \epsilon ) ( \forall u \in \mathcal{U} )</td>
</tr>
<tr>
<td>( q^* \uparrow ) as ( P \rightarrow P + \epsilon ) if ( r \downarrow )</td>
<td>Profit tax ( (1 - t)\Pi(q) )</td>
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</tbody>
</table>

Profit tax \( (1 - t)\Pi(q) \)

\( q^* \uparrow \) with \( t \) if \( r = \text{const and } R \uparrow \) or \( r \downarrow \) and \( R \uparrow \) \( r \downarrow \) and \( R = \text{const} \)

\( q^* \uparrow \) with \( t \) for all \( u \in \mathcal{U} \)

\[ r(z) := -\frac{u''(z)}{u'(z)}, \quad R(z) := zr(z) \]
**Investment in risky & safe assets**

*Arrow (1971)*

\[ Y(a) = A - a + (1 + X)a = A + aX \]

\( A = \text{initial wealth, } \quad Y = \text{final wealth.} \)

\( X = \text{rate of return of risky investment} \)

\( a = \text{amount invested in risky assets} \)

<table>
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<tr>
<th>EU</th>
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<tbody>
<tr>
<td>( a^* &gt; 0 ) iff ( EX &gt; 0 )</td>
<td>same</td>
</tr>
<tr>
<td>Wealth elasticity of cash balance ( m = A - a^* ) [ \frac{d m/d A}{m/A} \geq 1 ] if ( R \nearrow )</td>
<td>[ \frac{d m/d A}{m/A} \geq 1 ] ( \forall u \in U )</td>
</tr>
<tr>
<td>( X \rightarrow X + h \implies a^*(h) \nearrow )</td>
<td>same</td>
</tr>
<tr>
<td>( X \rightarrow (1 + h)X ) [ \implies a^<em>(h) = \frac{a^</em>}{1 + h} ]</td>
<td>same</td>
</tr>
</tbody>
</table>

*Tobin (1958)*
Investment in risky & safe assets

\[ Y(k) = k W_0 X + (1 - k) W_0 \rho \]

\( W_0 = \text{initial wealth} \)
\( X = \text{rate of return, risky asset} \)
\( \rho = \text{rate of return, safe asset} \)
\( k = \text{fraction invested in risky asset} \)

<table>
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<th>EU</th>
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<tr>
<td>If  ( 0 &lt; k^* &lt; 1 )</td>
<td>Conclusion holds for all  ( u \in U )</td>
</tr>
<tr>
<td>( \rho \uparrow \implies (1 - k^*) \uparrow )</td>
<td></td>
</tr>
<tr>
<td>if  ( r \uparrow ) or  ( R \leq 1 )</td>
<td></td>
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</table>
\( n \) states of nature
\[ p = (p_1, \ldots, p_n) \] their probabilities
\[ \bar{q}_i = \text{premium} \] for 1$ coverage in state \( i \)
\[ \bar{B} = \text{insurance budget} \]
\[ q_i = \frac{\bar{q}_i}{\sum_{j=1}^{n} \bar{q}_j} = \text{normalized premium} \]
\[ B = \frac{\bar{B}}{\sum_{j=1}^{n} \bar{q}_j} = \text{normalized budget} \]
\( x_i = \text{income} \) in state \( i \)
\( x = (x_1, \ldots, x_n) \) decision variable

\[ \sum_{i=1}^{n} q_i x_i = B \] budget constraint

The optimal value is

\[
I^* = \max_x \left\{ S_v([x, p]) : \sum_{i=1}^{n} q_i x_i = B \right\}
\]

\[
= \max_{x, \sum q_i x_i = B} \max_z \left\{ z + \sum_{i=1}^{n} p_i v(x_i - z) \right\}
\]

\[
= S_v([x^*, p])
\]

\( x^* = (x_i^*) \) is the optimal insurance coverage.
The optimal insurance coverage is

\[ I^* = \max \{ S_v([x, p]) : \sum_{i=1}^{n} q_i x_i = B \} \]

\[ = \max_{x, \sum q_i x_i = B} \max_{z} \{ z + \sum_{i=1}^{n} p_i v(x_i - z) \} \]

\[ = S_v([x^*, p]) \]

The optimal insurance coverage is

\[ x^*_i = B + \phi \left( \frac{q_i}{p_i} \right) - \sum_{j=1}^{n} q_j \phi \left( \frac{q_j}{p_j} \right) \]

where

\[ \phi = (v')^{-1} \]

Moreover,

\[ I^* = B - \sum q_i \phi \left( \frac{q_i}{p_i} \right) + \sum p_i v(\phi \left( \frac{q_i}{p_i} \right)) \]
The \( \phi \)-divergence functional

Csizsár (1978)

Given a convex function \( \phi : \mathbb{R}_+ \rightarrow \mathbb{R} \),
the \( \phi \)-divergence functional,

\[
I_\phi(p, q) := \sum_{i=1}^{n} q_i \phi \left( \frac{p_i}{q_i} \right)
\]

is a “distance function” on \( \mathbb{P}^n \).

\( \phi : \mathbb{R}_+ \rightarrow \mathbb{R} \) convex

\[\implies I_\phi(p, q) \text{ jointly convex in } p \text{ and } q.\]

Let \( \phi : \mathbb{R}_+ \rightarrow \mathbb{R} \) be convex.
Then for any \( p, q \in \mathbb{R}_+^n \),

\[
I_\phi(p, q) \geq \left( \sum_{i=1}^{n} q_i \right) \phi \left( \frac{\sum_{i=1}^{n} p_i}{\sum_{i=1}^{n} q_i} \right)
\]

If \( \phi \) is strictly convex, equality holds iff

\[
\frac{p_1}{q_1} = \frac{p_2}{q_2} = \cdots = \frac{p_n}{q_n}
\]
Adjoints

Let $\phi : \mathbb{R}_+ \to \mathbb{R}$ be convex, $\phi(1) = 0$. The adjoint of $\phi$ is

$$\phi^\diamond(t) := t \phi \left( \frac{1}{t} \right), \quad t > 0.$$ 

(a) $\phi^{\diamond \diamond} = \phi$.
(b) $\phi^\diamond(\cdot)$ is convex.
(c) $\phi^\diamond(1) = 0$
(d) For any $p, q \in \mathbb{R}_+^n$

$$I_\phi(p, q) = I_{\phi^\diamond}(q, p)$$

(e) $\phi^\diamond(t)$ is increasing for $t > 1$.

In general, $I_\phi(p, q) \neq I_\phi(q, p)$. Also $I_\phi(\cdot, \cdot)$ does not satisfy triangle inequality.

To get symmetry use:

$$I_\phi(p, q) + I_{\phi^\diamond}(p, q)$$
The Kullback-Leibler distance

\[ I_\phi(p, q) = \sum_{i=1}^{n} p_i \log \frac{p_i}{q_i} \]

from \( \phi(t) := t \log t, \quad t > 0, \)

\[ \phi^\diamond = t \left( \frac{1}{t} \log \frac{1}{t} \right) = -\log t \]

In particular, for \( p = \left( \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n} \right) \)

\[ I_{\phi^\diamond}(p, q) = -\sum_{i=1}^{n} q_i \log \frac{1}{nq_i} \]

\[ = \sum_{i=1}^{n} q_i \log q_i + \log n \]

The Hellinger distance

\[ I_\phi(p, q) = \sum_{i=1}^{n} (\sqrt{p_i} - \sqrt{q_i})^2 \]

from \( \phi(t) := \left( 1 - \sqrt{t} \right)^2, \quad t > 0 \)

\[ \phi^\diamond = \phi \]
For $\alpha > 1$ let,

$$\phi(t) := t^\alpha, \quad t > 0$$

$$I_{\phi}(p, q) = \sum_{i=1}^{n} (p_i)^\alpha (q_i)^{1-\alpha}$$

the $\alpha$-order entropy.

$$\phi^\diamond(t) = t^{1-\alpha}$$

### The $\chi^2$-distance

$$\phi(t) := (t - 1)^2, \quad t > 0, \quad I_{\phi}(p, q) = \sum_{i=1}^{n} \left( \frac{p_i^2 - q_i^2}{q_i} \right)$$

the $\chi^2$-distance.

$$\phi^\diamond(t) = \left( \sqrt{t} - \frac{1}{\sqrt{t}} \right)^2 = t + \frac{1}{t} - 2$$
The variation distance

\[
\phi(t) := |t - 1|, \quad t > 0,
\]

\[
I_\phi(p, q) = \sum_{i=1}^{n} |p_i - q_i|
\]

Indicator function

Let \(\alpha < 1 < \beta\) and define

\[
\phi(t) := \delta(t \mid [\alpha, \beta]) = \begin{cases} 
0 & \text{if } \alpha \leq t \leq \beta, \\
\infty & \text{otherwise}, 
\end{cases}
\]

indicator function of the interval \([\alpha, \beta]\),

\[
\phi^\diamond(t) = \delta\left(t \mid \left[\frac{1}{\beta}, \frac{1}{\alpha}\right]\right)
\]

\[
I_\phi(p, q) = \begin{cases} 
0 & \text{if } \alpha \leq \frac{p_i}{q_i} \leq \beta, \\
\infty & \forall i = 1, \ldots, n, \text{otherwise}.
\end{cases}
\]
For $f : \mathbb{R}^n \rightarrow \mathbb{R}$,
\[
 f^*(y) := \sup_x \{y \cdot x - f(x)\} \quad \text{convex conjugate}\n\]
\[
 f^*(y) := \inf_x \{y \cdot x - f(x)\} \quad \text{concave conjugate}\n\]
(a) $\phi$ convex $\implies (-\phi)^*(x) = -\phi^*(-x), \ \forall x$,
(b) $u$ concave $\implies (-u)^*(x) = -u^*(-x), \ \forall x$.

If $u : \mathbb{R} \rightarrow \mathbb{R}$ strictly $\nearrow$ & concave then
(a) $\text{dom } u^* \subset \mathbb{R}_+$
(b) $\text{dom } (u^{-1})^* \subset \mathbb{R}_+$
(c) $y \in \text{dom } (u^{-1})^* \iff \frac{1}{y} \in \text{dom } u^*$
(d) For all $y > 0$
\[
 (u^{-1})^*(y) = -y u^* \left( \frac{1}{y} \right) = -(u^*)^{\diamond}(y) \]

Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be convex. Then
(a) $\text{dom } \phi = \mathbb{R}_+ \implies \phi^*$ is nondecreasing
(b) $\text{dom } \phi = \mathbb{R}_{++} \implies \phi^*$ is increasing
For $h : \mathbb{R}^n \to \mathbb{R}$ define $h_+ : \mathbb{R}^n \to \mathbb{R}$ by

$$h_+(y) := \sup_{z \in \mathbb{R}, x \in \mathbb{R}^n} \{ z + h(x - ze) \}$$

Then for any $y \in \mathbb{R}^n$ with $\sum y_i \neq 0$,

$$h_+(y) = -h_\star \left( \frac{y}{\sum y_i} \right)$$

If $h$ is strictly concave and $C^{(1)}$,

$$z^* = -v^T u^* = -\frac{y^T}{e \cdot y} (\nabla h)^{-1} \left( \frac{y}{e \cdot y} \right)$$

$$x^* = u^* + z^* e = \left( I - \frac{ey^T}{e^T y} \right) (\nabla h)^{-1} \left( \frac{y}{e \cdot y} \right)$$

$$= P_{e\perp} (\nabla h)^{-1} \left( \frac{y}{e \cdot y} \right)$$

$P_{e\perp}$ := orthogonal projection $\perp e$.

$$-z^* e = \frac{ey^T}{e^T y} (\nabla h)^{-1} \left( \frac{y}{e \cdot y} \right) = P_e (\nabla h)^{-1} \left( \frac{y}{e \cdot y} \right)$$
Extremal principles for RCE, EU

If \( u : \mathbb{R} \rightarrow \mathbb{R} \) is strictly \( \nearrow \) & concave, then

\[
\phi := -u^*
\]

is convex and \( \phi : \mathbb{R}_+ \rightarrow \mathbb{R} \). Also

\[
\phi^\diamond = (u^{-1})^*
\]

Conversely, for convex \( \phi : \mathbb{R}_+ \rightarrow \mathbb{R} \),

\[
u(t) := -\phi^*(-t)
\]

is concave.

For any RV \( X = [x, p] \),

\[
S_u ([x, p]) = \inf_{q \in \mathbb{P}^n} \left\{ I_\phi(q, p) + \sum_{i=1}^{n} q_i x_i \right\}
\]

\[
= \inf_{q \in \mathbb{P}^n} \left\{ I_{\phi^\diamond}(p, q) + \sum_{i=1}^{n} q_i x_i \right\}
\]

\[
E_u ([x, p]) = \inf_{q \in \mathbb{R}_+^n} \left\{ I_\phi(q, p) + \sum_{i=1}^{n} q_i x_i \right\}
\]
Duality

\[ S_u([x, p]) = \inf_{q \in \mathbb{P}^n} \left\{ I_\phi(q, p) + \sum_{i=1}^{n} q_i x_i \right\} \]

The RCE, in LHS, is the optimal value of

\((P)\) \quad \sup \{z : \text{”} z \leq X \text{”} \}

in the sense of recourse optimization.

The RHS is the “dual” of \((P)\),

\((D)\) \quad \inf \{z : \text{”} z \geq X \text{”} \}

using a stochastic penalty function \(P(\cdot)\) to “enforce” the stochastic constraint

\text{“} z \geq X \text{”} \\

\((D1)\) \quad \inf \{z + P(z)\}
Given the RV $X = [x, p]$ and $z$, define

$$R\{z\} := \left\{ q \in P^n : z \geq \sum_i q_i x_i \right\},$$

representing the RV’s $\{Z := [x, q] : q \in R\{z\}\}$ supported as $X$, satisfying $z \geq X$ “in the mean”. $P(z)$ is the penalty

$$P(z) := “\text{dist”} (p, R\{z\}) = \inf_{q \in R\{z\}} “\text{dist”} (q, p)$$

and “dist” is induced by the $\phi$–divergence,

$$P(z) = \inf_{q \in R\{z\}} I\phi(q, p)$$

$$\inf_{z} \{z + I\phi(q, p)\} =$$

$$\inf_{q \in R\{z\}} \inf_{z \geq \sum q_i x_i} \{z + I\phi(q, p)\}$$

$$= \inf_{q \in P^n} \{I\phi(q, p) + \sum q_i x_i\}$$
Extremal principles for $\phi$–divergence

Let $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}$ convex, $\text{dom } \phi \subset \mathbb{R}_{++}$. Define

$$u(t) := -\phi^*(-t)$$

Then for any $p, q \in \mathbb{P}^n$,

$$I_\phi(q, p) = \sup_{x \in \mathbb{R}^n, \ q \cdot x = 0} S_u([x, p])$$

Furthermore, if $\text{dom } \phi = \mathbb{R}_{++}$, define the function

$$v(x) := (\phi^*)^{-1}(x),$$

Then

$$I_\phi(p, q) = \sup_{x \in \mathbb{R}^n, \ q \cdot x = 0} S_v([x, p])$$

For all $p \in \mathbb{P}^n, q \in \mathbb{R}_{++}^n$,

$$I_\phi(q, p) = \sup_{x \in \mathbb{R}^n} \left\{ \sum_{i=1}^n p_i u(x_i) - \sum_{i=1}^n q_i x_i \right\}$$