SIR Models continued

Using $R = N - S - I$ we obtain the simplified system

$$\dot{S} = -aSI + c(N - S - I)$$
$$\dot{I} = (aS - b)I$$

This has Jacobian

$$J(S, I) = \begin{pmatrix} -aI - c & -aS - c \\ aI & aS - b \end{pmatrix}$$

Stability

The Steady state $S = N$, $I = 0$ has

$$J(N,0) = \begin{pmatrix} -c & -aN - c \\ 0 & aN - b \end{pmatrix}$$

which implies $\lambda_1 + \lambda_2 = -c + (aN - b)$, and $\lambda_1\lambda_2 = -c(aN - b)$. This implies it is unstable if $aN - b > 0$ and stable if $aN - b < 0$.

Stability

The steady state $S = \frac{b}{a}$, $I = \frac{c(N-b/a)}{b+c}$ requires $N - b/a > 0$ for a nonzero infected population.

$$J = \begin{pmatrix} -\frac{ac(N-b/a)}{b+c} - c & -b - c \\ \frac{ac(N-b/a)}{b+c} & 0 \end{pmatrix}$$

which implies $\lambda_1 + \lambda_2 = -\frac{ac(N-b/a)}{b+c} - c$, and $\lambda_1\lambda_2 = ac(N - b/a)$. Since $N - b/a > 0$ we have $\lambda_1 + \lambda_2 < 0$ and $\lambda_1\lambda_2 > 0$. Hence when feasible, this steady state is also stable.

Calculating R_0

At equilibrium, we have that the fraction of the population susceptible to the disease $x^* = \frac{b/a}{N}$ has

$$x^* R_0 = 1$$

Calculating R_0

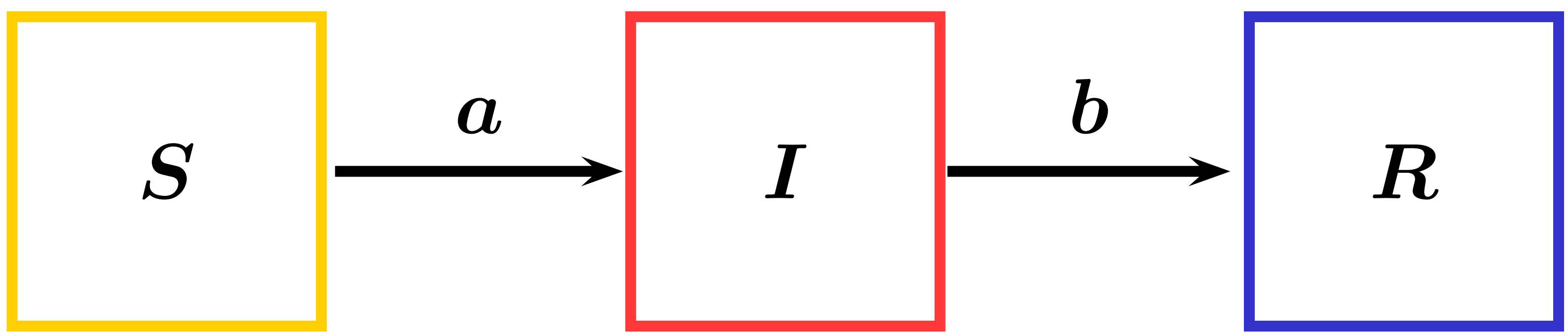
Another way of determining R_0 is through the initial rise in infectives. Since $\dot{I} = b(R_0 - 1)I$ initially, then determining the growth rate of infectives λ implies

$$\begin{aligned} R_0 &= \frac{\lambda}{b} + 1 \\ &= \lambda t_{ave} + 1 \end{aligned}$$

where $t_{ave} = 1/b$ is the average time infected.

SIR Model

If $c = 0$ so that no recovered revert to susceptible, then we have an SIR model (no extra S).



SIR Example

In a boys' boarding school, containing 763 boys, in Britain 1978, one boy with the flu started an epidemic. The following table shows the course of the epidemic.

t (days)	0	3	4	5	6	7	
$I(t)$	1	22	80	228	298	268	
t	8	9	10	11	12	13	14
$I(t)$	230	200	120	82	24	6	4

Determine the parameters for this epidemic and R_0.

Vaccination

The community will be immune to infection if $\boldsymbol{R_0 = \frac{Na}{b} < 1}$. This is called “herd immunity" since although individuals may be infected, the herd will be safe.

Vaccination against a disease can be successful without making everyone immune. It does so by utilizing herd immunity. By reducing the size of the population against which the disease can act, to below the threshhold level, the disease will not produce an epidemic.

Vaccination

What fraction p of the population needs to be vaccinated for herd immunity to apply against a disease?

Vaccination to this extent reduces the population the disease can infect from N to $(1-p)N$. So we need

$$\frac{a(1-p)N}{b} = (1-p)R_0 < 1$$